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HIGHER LEVEL WZW SECTORS FROM FREE FERMIONS

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Abstract.
We introduce a gauge group of internal symmetries of an ambient algebra as a new tool for investigating the superselection structure of WZW theories and the representation theory of the corresponding affine Lie algebras. The relevant ambient algebra arises from the description of these conformal field theories in terms of free fermions. As an illustration we analyze in detail the so(N) WZW theories at level two. In this case there is actually a homomorphism from the representation ring of the gauge group to the WZW fusion ring, even though the level-two observable algebra is smaller than the gauge invariant subalgebra of the field algebra.

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1 Introduction

While a wealth of information about WZW theories has been obtained by analyzing these conformal field theories with the help of the unbounded operators which generate their Virasoro and affine Lie algebra structures, much less is known about the superselection structure of WZW theories as described in terms of local algebras of bounded operators. In comparison with higher-dimensional relativistic quantum field theories, some difficulties arise in these models as a consequence of the fact that the quantum symmetry which governs the superselection structure is not a gauge group in the sense of Doplicher, Haag and Roberts (DHR [1]). So far no generally accepted description of this quantum symmetry is available. Accordingly, the analysis of WZW models in the framework of algebraic field theory has been confined to the case of level one of the relevant affine Lie algebras [2, 3] or to simple currents [4], i.e. sectors with unit statistical dimension. [1] Here we report on ideas which allow to deal also with more complicated situations.

The purpose of this paper is twofold. First, we would like to find a convenient substitute for the DHR gauge group in low-dimensional field theories. We do *not* require that this substitute plays the rôle of the full quantum symmetry of the theory, i.e. the gauge invariant fields need not coincide with the observables of the theory under consideration, so that the gauge group does not directly describe the superselection structure. Rather, we only demand that it supplies a tool for examining this structure, which when combined with other information allows to characterize the sectors at least to a large extent. In the specific case of WZW theories, the required additional information comes from the representation theory of affine Lie algebras. In this case we succeed in identifying a symmetry group which satisfies the required property.

Our second goal is to get a new handle on certain aspects of the representation theory of affine Lie algebras, in particular to obtain simple formulæ for the characters of irreducible highest weight modules. While there exists a closed expression, the well known Weyl–Kac formula [10], for all these characters, it is often difficult to evaluate because it involves a summation over the Weyl group of the horizontal subalgebra $\bar{\mathfrak{g}}$ of the affine algebra $\mathfrak{g}$. Therefore one often prefers to have formulæ which are better controllable, say in terms of infinite sums or products that are easy to handle with algebraic manipulation programs. For instance, in the case of the classical series of simple Lie algebras $\bar{\mathfrak{g}}$ one would like to have a form of the characters which has a simple functional dependence on the rank of $\bar{\mathfrak{g}}$.

Simple character formulæ of this type are in particular known for the level-one modules of many algebras, owing to the fact that these modules can be realized in terms of free bosons (when $\bar{\mathfrak{g}}$ is simply laced) or free fermions (when $\bar{\mathfrak{g}}$ is so(N) or su(N)). Now all irreducible modules at arbitrary positive integral level can be obtained as submodules of tensor products of level-one modules. Therefore in principle the realization through free fields can also be exploited at higher level. The problem with this approach is that in general it is extremely difficult to identify the irreducible submodules in tensor products. In particular, the branching 'coefficients' for tensor products of affine Lie algebra modules are not numerical constants, but have a functional dependence on (part of) the Cartan subalgebra of $\mathfrak{g}$. More specifically, when the branching rules are expressed in terms of the characters of the modules, these branching functions constitute characters of the

[1] Similar remarks apply to the work on other conformal field theories, compare e.g. [5, 6, 7, 8]. For an approach which addresses general conformal field theories, see [9].

observable algebra of the coset conformal field theory

$$\mathcal{C}os \simeq (\mathfrak{g}_{\text{level } 1})^{\oplus k^\vee} / \mathfrak{g}_{\text{level } k^\vee}$$

(see e.g. [11]). These coset characters are sometimes known even when the branching rules are not, e.g. when the coset theory can be described as a conformal field theory also in a different manner.

Our approach to the problems outlined above is based on the following idea. We define a field algebra $\mathfrak{F}$ which is essentially the CAR algebra of $k^\vee$ species of free fermions acting on a big Fock space which is the $k^\vee$-fold tensor product of the Fock space of the level-one theory. (To avoid technical complications, for the purposes of this paper we restrict our attention to the Neveu–Schwarz sector of the fermions.) [2] This algebra comes with a natural symmetry group $\mathrm{O}(k^\vee)$ (for real fermions, respectively $\mathrm{U}(k^\vee)$ for complex fermions) which, roughly speaking, rotates the different fermion species into each other. It is this group $\mathrm{O}(k^\vee)$ (respectively $\mathrm{U}(k^\vee)$) which we propose as a substitute for the gauge group in the DHR sense. Accordingly we introduce an 'intermediate' observable algebra $\mathfrak{A}$, to which we will refer as the *gauge invariant fermion algebra*, which is defined as the gauge invariant subalgebra of the field algebra $\mathfrak{F}$. The observables of the level-$k^\vee$ WZW theory are naturally gauge invariant, and hence they do not make transitions between the sectors of $\mathfrak{A}$. Let us denote by $\mathfrak{A}_{\mathrm{WZW}}$ the observable algebra of *bounded* operators which is associated to the WZW model; it can be constructed from the positive energy representations of the corresponding loop group acting on the big Fock space. The algebra $\mathfrak{A}_{\mathrm{WZW}}$ contains the bounded functions of local current operators, and the irreducible representation spaces of the positive energy representations are precisely the highest weight modules of the chiral symmetry algebra of the WZW theory. The latter is given by the semidirect sum of the untwisted affine Lie algebra $\widehat{\mathrm{so}}(N)$ at level $k^\vee$ and the Virasoro algebra that is associated to $\widehat{\mathrm{so}}(N)$ by the Sugawara construction (hence in particular it consists of *unbounded* operators).

Just like in the DHR situation, the sectors of the gauge invariant fermion algebra $\mathfrak{A}$ can be described with the help of the representation theory of the gauge group. However, except for level one, these are different from the sectors of the WZW theory, because the observables $\mathfrak{A}_{\mathrm{WZW}}$ of the WZW theory do not exhaust the invariants of the gauge group, i.e. we have the proper inclusions

$$\mathfrak{A}_{\mathrm{WZW}} \subset \mathfrak{A} \subset \mathfrak{F}\,.$$

Nevertheless a lot of information about the decomposition of tensor products of level-one modules into modules of the level-$k^\vee$ chiral algebra can be obtained by decomposing the big Fock space into the irreducible sectors of $\mathfrak{A}$. This is possible because the Virasoro algebra of the coset conformal field theory $(\mathfrak{g}_{\text{level } 1})^{\oplus k^\vee} / \mathfrak{g}_{\text{level } k^\vee}$ is gauge invariant as well, so that we can combine this decomposition with information on the representation theory of the coset Virasoro algebra.

In this paper we will examine in detail a non-trivial theory which already displays the generic features of the superselection structure, but can still be managed without having to delve into too many technicalities. Namely, we will treat the case of two species of real

[2] Instead of free fermions, one might also use free bosons to implement our ideas. However, technically these are more difficult to handle because in order to deal with genuine conformal fields one must study vertex operators. In contrast, free fermions are proper conformal fields already themselves.

fermions, corresponding to the gauge group O(2). The simplicity of this example can be regarded as reflecting the fact that the representations of O(2) are at most two-dimensional. Furthermore, information about this theory is also available from other sources, namely [12] certain conformal embeddings of affine Lie algebras, so that we can cross-check our results. The study of more complicated theories will be left to future work.

Our paper ist organized as follows. In the next three sections we describe the algebraic aspects of free fermions: the CAR algebra (section 2), the associated gauge group $\mathrm{O}(k^\vee)$ (section 3), and some specific features of the level-two gauge group O(2) (section 4). Afterwards we provide some basic information about the various Lie algebraic and conformal field theory structures that will be employed: the simple Lie algebra $\mathrm{so}(N)$ (section 5), the affine Lie algebra $\widehat{\mathrm{so}}(N)$ and the spectrum of the associated WZW theory at levels one and two (section 6), and the $\mathbb{Z}_2$-orbifold conformal field theories with conformal central charge $c=1$ (section 7). Then we proceed to the analysis of the decomposition of tensor products of level-one $\widehat{\mathrm{so}}(N)$-modules. First the highest weight vectors of $\widehat{\mathrm{so}}(N)$ at level two and of the coset Virasoro algebra (section 8) are identified. We are then in a position to compute the characters of the sectors of $\mathfrak{A}$ (subsection 9.1) and of $\widehat{\mathrm{so}}(N)_2$ (subsection 9.2 and 9.3). In the final section 10 we summarize our results on the tensor product decomposition, and we also remark on implications of the representation theory of the gauge group for the fusion rules of the WZW theory.

2 The CAR algebra

We consider a separable Hilbert space $\mathcal{K}$ endowed with an anti-unitary involution Γ (complex conjugation), $\Gamma^2 = id$, which obeys

$$\langle \Gamma f, \Gamma g\rangle = \langle g, f\rangle \tag{2.1}$$

for all $f,g\in\mathcal{K}$. The selfdual canonical anti-commutation relation (CAR) algebra $\mathcal{C}(\mathcal{K},\Gamma)$ corresponding to a single free fermion is defined to be the C^*-norm closure of the algebra that is generated by the range of a linear mapping $B: f\mapsto B(f)$ of the Hilbert space which possesses the property that

$$\{B(f)^*, B(g)\} = \langle f,g\rangle\,\mathbf{1}\,, \qquad B(f)^* = B(\Gamma f) \tag{2.2}$$

holds for all $f,g\in\mathcal{K}$ (see e.g. [13]).

By definition, a quasi-free state ω of $\mathcal{C}(\mathcal{K},\Gamma)$ fulfills

$$\omega(B(f_1)\cdots B(f_{2n+1})) = 0\,, \tag{2.3}$$

$$\omega(B(f_1)\cdots B(f_{2n})) = (-1)^{n(n-1)/2}\sum_\sigma \mathrm{sign}(\sigma)\prod_{j=1}^n \omega(B(f_{\sigma(j)})B(f_{\sigma(n+j)})) \tag{2.4}$$

for all $n\in\mathbb{N}$, where the sum runs over all permutations $\sigma\in\mathcal{S}_{2n}$ with the property

$$\sigma(1)<\sigma(2)<\ldots<\sigma(n) \qquad \text{and} \qquad \sigma(j)<\sigma(j+n) \quad \text{for } j=1,2,\ldots,n\,. \tag{2.5}$$

Quasi-free states are completely characterized by their two point functions. Moreover, the formula

$$\omega(B(f)^*B(g)) = \langle f, Sg\rangle \tag{2.6}$$

provides a one-to-one correspondence between the set of quasi-free states of $\mathcal{C}(\mathcal{K},\Gamma)$ and the subset

$$\mathcal{Q}(\mathcal{K},\Gamma) := \{S\in\mathfrak{B}(\mathcal{K}) \,|\, S=S^*,\ 0\le S\le \mathbf{1},\ S+\Gamma S\Gamma=\mathbf{1}\} \tag{2.7}$$

of $\mathfrak{B}(\mathcal{K})$ (the set of bounded operators on $\mathcal{K}$). It is therefore convenient to denote the quasi-free state characterized by equation (2.6) by ω_S. The projections in $\mathcal{Q}(\mathcal{K},\Gamma)$ are called basis projections or polarizations. For a basis projection P, the state ω_P is pure and is called a Fock state. The corresponding GNS representation $(\mathcal{H}_P,\pi_P,|\Omega_P\rangle)$ is irreducible; it is called the Fock representation. The space $\mathcal{H}_P$ can be canonically identified with the antisymmetric Fock space $\mathcal{F}_-(P\mathcal{K})$.

Let us now specialize to the Hilbert space

$$\mathcal{K}=L^2(S^1;\mathbb{C}^N)\equiv L^2(S^1)\otimes\mathbb{C}^N\,, \tag{2.8}$$

which corresponds to a fermion living on the circle S^1 and carrying the N-dimensional vector representation of the Lie algebra so(N). The involution Γ is given by component-wise complex conjugation. We introduce a (Fourier) orthonormal basis

$$\{e^i_r \mid r\in\mathbb{Z}+\tfrac12,\ i=1,2,\dots,N\} \tag{2.9}$$

of $\mathcal{K}$ by the definition

$$e^i_r := e_r\otimes u^i \qquad \text{for}\quad r\in\mathbb{Z}+\tfrac12\,,\ i=1,2,\dots,N\,, \tag{2.10}$$

where $e_r\in L^2(S^1)$ are defined by $e_r(z)=z^r$ (with $z=\mathrm{e}^{\mathrm{i}\varphi}$, $-\pi<\varphi\le\pi$), and where u^i denote the canonical unit vectors of $\mathbb{C}^N$. The Neveu–Schwarz operator $P_{\rm NS}\in\mathcal{Q}(\mathcal{K},\Gamma)$ is then by definition the basis projection

$$P_{\rm NS} := \sum_{i=1}^{N}\ \sum_{r\in\mathbb{N}_0+1/2} |e^i_{-r}\rangle\,\langle e^i_{-r}|\,. \tag{2.11}$$

The GNS representation associated to the Fock state $\omega_{P_{\rm NS}}$ provides the Fock space $\mathcal{H}_{\rm NS}$ which decomposes into the basic and the vector module of $\widehat{\rm so}(N)$ at level one. (In this paper we only discuss the Neveu–Schwarz sector. The Ramond sector, in which a Fourier basis with integer powers of z appears will not be considered here. It could be analyzed by the same methods, but the technical details are considerably more involved.)

We are interested in the theory that is obtained when one considers an arbitrary number $k^\vee$ of Neveu–Schwarz fermions of the type described above. Thus in addition to the so(N) index i the fermion modes will now be labelled by a 'flavor' index q which takes values in $\{1,2,\dots,k^\vee\}$. To describe this theory, we define

$$\hat{\mathcal{K}} := \mathcal{K}\otimes\mathbb{C}^{k^\vee},\quad \hat\Gamma := \Gamma\otimes\Gamma_{k^\vee}\quad\text{and}\quad \hat P_{\rm NS} := P_{\rm NS}\otimes \mathbb{1}_{k^\vee}\,, \tag{2.12}$$

where $\Gamma_{k^\vee}$ denotes the canonical complex conjugation in $\mathbb{C}^{k^\vee}$. Further, for any $f\in\mathcal{K}$ we define the elements

$$B^q(f) := B(f\otimes v^q)\,,\qquad q=1,2,\dots,k^\vee\,, \tag{2.13}$$

of $\mathcal{C}(\hat{\mathcal{K}},\hat\Gamma)$, where v^q denote the canonical unit vectors of $\mathbb{C}^{k^\vee}$. By $(\hat{\mathcal{H}}_{\rm NS},\hat\pi_{\rm NS},|\hat\Omega_{\rm NS}\rangle)$ we denote the GNS representation associated to the Fock state $\omega_{\hat P_{\rm NS}}$ of $\mathcal{C}(\hat{\mathcal{K}},\hat\Gamma)$; we will refer to $\hat{\mathcal{H}}_{\rm NS}$ as the 'big Fock space'. We then define the Fourier modes

$$b^{i;q}_r := \hat\pi_{\rm NS}(B^q(e^i_r)) \tag{2.14}$$

for $i = 1, 2, \ldots, N$, $q = 1, 2, \ldots, k^\vee$ and $r \in \mathbb{Z} + \frac{1}{2}$. The Fourier modes $b_r^{i;q}$ generate a CAR algebra with relations

$$\{b_r^{i;p}, b_s^{j;q}\} \;=\; \delta_{p,q}\, \delta_{i,j}\, \delta_{r,-s}\, \mathbf{1}\,. \tag{2.15}$$

The modes $b_r^{i;q}$ with positive index r act as annihilation operators in $\hat{\mathcal{H}}_{\rm NS}$, i.e. for all $q = 1, 2, \ldots, k^\vee$ and all $i = 1, 2, \ldots, N$ we have

$$b_r^{i;q}\, |\hat\Omega_{\rm NS}\rangle = 0 \qquad \text{for } r \in \mathbb{N}_0 + \tfrac{1}{2}\,. \tag{2.16}$$

3 The gauge group $\mathrm{O}(k^\vee)$

Field and observable algebras of the fermion theory are described as follows. Choose a point $\zeta \in S^1$ on the circle and denote by $\mathcal{J}_\zeta$ the set of those open intervals $I \subset S^1$ whose closures do not contain ζ. For $I \in \mathcal{J}_\zeta$ let $\mathcal{K}(I)$ be the subspace of functions having support in I. Correspondingly, define $\hat{\mathcal{K}}(I) = \mathcal{K}(I) \otimes \mathbb{C}^{k^\vee}$. The local field algebras $\mathfrak{F}(I)$ are then defined to be the von Neumann algebras

$$\mathfrak{F}(I) = \hat\pi_{\rm NS}(\mathcal{C}(\hat{\mathcal{K}}(I), \hat\Gamma))'' \tag{3.1}$$

(the prime denotes the commutant in $\mathfrak{B}(\hat{\mathcal{H}}_{\rm NS})$), and the global field algebra $\mathfrak{F}$ is the C^*-algebra that is defined as the norm closure of the union of the local algebras,

$$\mathfrak{F} = \overline{\bigcup_{I \in \mathcal{J}_\zeta} \mathfrak{F}(I)}\,. \tag{3.2}$$

Now the group $\mathrm{O}(k^\vee)$ acts in the natural way on the multiplicity space $\mathbb{C}^{k^\vee}$ in (2.12), and this extends canonically to an action on $\mathcal{C}(\hat{\mathcal{K}}, \hat\Gamma)$ by Bogoliubov automorphisms. Moreover, by construction, these automorphisms leave the Fock state $\omega_{\hat P_{\rm NS}}$ invariant. Hence we obtain a unitary representation U of $\mathrm{O}(k^\vee)$ in $\hat{\mathcal{H}}_{\rm NS}$. Also its action respects the local structure (3.1), and thus $\mathrm{O}(k^\vee)$ can be regarded as a substitute for the gauge group in the sense of Doplicher, Haag and Roberts [1]. This a subgroup of the automorphism group of $\mathfrak{F}(I)$ respectively $\mathfrak{F}$ such that the observables are precisely the gauge invariant fields. Therefore the local observable algebras $\mathfrak{A}(I)$ and the global (or quasi-local) observable algebra $\mathfrak{A}$ are defined as $\mathrm{O}(k^\vee)$-invariant part of the field algebras,

$$\mathfrak{A}(I) = \mathfrak{F}(I) \cap U(\mathrm{O}(k^\vee))' \tag{3.3}$$

and

$$\mathfrak{A} = \overline{\bigcup_{I \in \mathcal{J}_\zeta} \mathfrak{A}(I)}\,. \tag{3.4}$$

In the level-one case, the intermediate algebra $\mathfrak{A}$ coincides with the observable algebra $\mathfrak{A}_{\rm WZW}$ of the WZW theory, so that the representation theory of $\mathfrak{A}$ reproduces precisely the sectors of the observable algebra $\mathfrak{A}_{\rm WZW}$; furthermore, the DHR product of the sectors of the gauge (i.e. $\mathrm{O}(1) \equiv \mathbb{Z}_2$) invariant fermion algebra $\mathfrak{A}$ that is obtained by composing localized endomorphisms of $\mathfrak{A}$ provides the WZW fusion rules [3]. In contrast, at higher level the algebra $\mathfrak{A}$ no longer coincides with the observable algebra $\mathfrak{A}_{\rm WZW}$ of the WZW theory. Indeed we will see that already at level $k^\vee = 2$ each irreducible $\mathfrak{A}$-sector is highly reducible under the action of the observable algebra $\mathfrak{A}_{\rm WZW}$. Nevertheless, owing to

$\mathfrak{A}_{\rm WZW} \subset \mathfrak{A}$ the representation theory of $\mathfrak{A}$ is crucial for our analysis of the decomposition of the big Fock space into tensor products of highest weight modules of the level-2 chiral algebra and of the coset Virasoro algebra.

For the construction of the highest weight vectors within the $\mathfrak{A}$-sectors it is convenient to work with the unbounded operators of $\widehat{\rm so}(N)$ (instead of the bounded elements of $\mathfrak{A}_{\rm WZW}$) and of the Virasoro algebra that is associated to $\widehat{\rm so}(N)$ (at fixed level) by the Sugawara formula. The generators of this Virasoro algebra, i.e. the Laurent modes of the energy-momentum tensor of the WZW theory, will be denoted by L_m. Also, we denote by $L_m^{\rm NS}$ the Laurent components of the canonical energy-momentum tensor of the fermion theory in the Neveu–Schwarz representation, i.e.

$$L_m^{\rm NS} = \sum_{q=1}^{k^\vee} L_m^{(q)} \qquad \text{with} \quad L_m^{(q)} = -\frac{1}{2} \sum_{i=1}^{N} \sum_{r\in\mathbb{Z}+1/2} r \,{:}b_r^{i;q} b_{m-r}^{i;q}{:}\ . \tag{3.5}$$

Thus in particular

$$L_0^{(q)} = \sum_{i=1}^{N} \sum_{r\in\mathbb{N}_0+1/2} r\, b_{-r}^{i;q} b_r^{i;q}\,. \tag{3.6}$$

The Bogoliubov automorphisms act as rotations on the ‘flavor’ index q of the fermions. As a consequence, they leave expressions of the form

$$\sum_{q=1}^{k^\vee} B^q(f)\, B^q(g) \qquad (f,g \in \mathcal{K}) \tag{3.7}$$

invariant. In particular, owing to the summation on q in the bilinear expression (3.5), the Virasoro generators $L_m^{\rm NS}$ are $\mathrm{O}(k^\vee)$-invariant. This implies that the coset Virasoro operators

$$L_m^{\rm c} := L_m^{\rm NS} - L_m \tag{3.8}$$

are gauge invariant as well.

4 The gauge group at level 2

Let us now specialize to the case $k^\vee = 2$. Thus we consider the situation

$$\hat{\mathcal{K}} = \mathcal{K} \oplus \mathcal{K}\,, \quad \hat\Gamma = \Gamma \oplus \Gamma \quad \text{and} \quad \hat P_{\rm NS} = P_{\rm NS} \oplus P_{\rm NS}\,. \tag{4.1}$$

Then the transformations

$$\begin{aligned} \gamma_t(B^1(f)) &:= \cos(t)\, B^1(f) - \sin(t)\, B^2(f)\,, \\ \gamma_t(B^2(f)) &:= \sin(t)\, B^1(f) + \cos(t)\, B^2(f) \end{aligned} \tag{4.2}$$

for $t\in\mathbb{R}$ and

$$\eta(B^1(f)) := B^1(f)\,, \qquad \eta(B^2(f)) := -B^2(f)\,. \tag{4.3}$$

define Bogoliubov automorphisms of $\mathcal{C}(\hat{\mathcal{K}},\hat\Gamma)$ generating the group $\mathrm{O}(2)$. The invariance of the Fock state $\omega_{\hat P_{\rm NS}}$ reads now

$$\omega_{\hat P_{\rm NS}} \circ \gamma_t = \omega_{\hat P_{\rm NS}} = \omega_{\hat P_{\rm NS}} \circ \eta\ , \tag{4.4}$$

and there is a unitary (strongly continuous) representation U of $O(2)$ by certain implementers $U(\gamma_t), U(\eta) \in \mathfrak{B}(\hat{\mathcal{H}}_{\rm NS})$ which satisfy

$$U(\gamma_t)\,|\hat{\Omega}_{\rm NS}\rangle = |\hat{\Omega}_{\rm NS}\rangle = U(\eta)\,|\hat{\Omega}_{\rm NS}\rangle\,, \tag{4.5}$$

and the action of γ_t and η extends to $\mathfrak{B}(\hat{\mathcal{H}}_{\rm NS})$.

The inequivalent finite-dimensional irreducible representations of O(2) are the following. Besides the identity Φ_0 with $\Phi_0(\cdot) = 1$ and another one-dimensional representation Φ_J with

$$\Phi_J(\gamma_t) = 1\,, \qquad \Phi_J(\eta) = -1\,, \tag{4.6}$$

there are only two-dimensional representations $\Phi_{[m]}$ with $m = 1, 2, \ldots$; their representation matrices are

$$\Phi_{[m]}(\gamma_t) = \begin{pmatrix} {\rm e}^{{\rm i}mt} & 0 \\ 0 & {\rm e}^{-{\rm i}mt} \end{pmatrix}, \qquad \Phi_{[m]}(\eta) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.7}$$

The tensor product decompositions of these representations read

$$\begin{aligned}
&\Phi_J \times \Phi_J = \Phi_0\,, \qquad\qquad \Phi_J \times \Phi_{[m]} = \Phi_{[m]}\,,\\
&\Phi_{[m]} \times \Phi_{[n]} = \Phi_{[|m-n|]} \oplus \Phi_{[m+n]} \qquad \text{for } m \neq n\,,\\
&\Phi_{[n]} \times \Phi_{[n]} = \Phi_0 \oplus \Phi_J \oplus \Phi_{[2n]}\,.
\end{aligned} \tag{4.8}$$

Employing the results of [1], it then follows that the Hilbert space $\hat{\mathcal{H}}_{\rm NS}$ decomposes into irreducible sectors of the global observable algebra $\mathfrak{A}$ as

$$\hat{\mathcal{H}}_{\rm NS} = \mathcal{H}_0 \oplus \mathcal{H}_J \oplus \bigoplus_{m=1}^{\infty} (\mathcal{H}_{[m]} \otimes H_{[m]})\,. \tag{4.9}$$

Here $\mathcal{H}_0$, $\mathcal{H}_J$ and $\mathcal{H}_{[m]}$ carry mutually inequivalent irreducible representations of $\mathfrak{A}$; vectors in $\mathcal{H}_0, \mathcal{H}_J$ transform according to the two inequivalent one-dimensional irreducible representations Φ_0 and Φ_J of the gauge group O(2), respectively, and the $H_{[m]} \simeq \mathbb{C}^2$ carry the inequivalent two-dimensional irreducible O(2)-representations $\Phi_{[m]}$. Later we will also use the notation

$$\mathcal{H}_{[m]} \otimes H_{[m]} = \mathcal{H}^+_{[m]} \oplus \mathcal{H}^-_{[m]}\,, \tag{4.10}$$

where by definition, $U(\gamma_t)$ acts on $\mathcal{H}^\pm_{[m]}$ as multiplication with ${\rm e}^{\pm {\rm i}mt}$.

5 The simple Lie algebra so(N)

Normal-ordered bilinears in the free fermions modes that were introduced in section 2 realize the affine Kac–Moody algebra $\widehat{\rm so}(N)$ at level $k^\vee$. To describe this realization of $\widehat{\rm so}(N)$, we first need to collect some information about the simple Lie algebra so(N) which is canonically embedded in $\widehat{\rm so}(N)$. We denote by ℓ the rank of the Lie algebra so(N), i.e. $\ell = N/2$ and $\ell = (N-1)/2$ for even and odd N, respectively.

We use the notation $T^{i,j} := {\rm i}\,(E^{i,j} - E^{j,i})$ for $i, j = 1, 2, \ldots, N$, where $E^{i,j}$ are the matrix units, which have entries $(E^{i,j})_{k,l} = \delta_{i,k}\delta_{j,l}$. The matrices $T^{i,j}$ satisfy

$$[T^{i,j}, T^{k,l}] = {\rm i}\,(\delta_{j,k}T^{i,l} + \delta_{i,l}T^{j,k} - \delta_{j,l}T^{i,k} - \delta_{i,k}T^{j,l})\,. \tag{5.1}$$

Next we introduce the combinations

$$H^j := T^{2j-1,2j} \quad \text{for } j = 1, 2, \ldots, \ell \tag{5.2}$$

and

$$E^j_\pm := \pm t^{j,j+1}_{\pm,\mp} \;\text{ for } j = 1, 2, \ldots, \ell-1\,, \qquad E^\ell_\pm := \begin{cases} \pm t^{\ell-1,\ell}_{\pm,\pm} & \text{for } N = 2\ell\,, \\ \pm t^\ell_\pm & \text{for } N = 2\ell+1\,, \end{cases} \tag{5.3}$$

where

$$\begin{aligned} t^{i,j}_{\varepsilon,\eta} &:= \tfrac{1}{2}\,(\varepsilon T^{2i,2j-1} + \eta T^{2i-1,2j}) + \tfrac{\mathrm{i}}{2}\,(T^{2i-1,2j-1} - \varepsilon\eta T^{2i,2j})\,, \\ t^j_\varepsilon &:= -\frac{1}{\sqrt{2}}\,(\varepsilon T^{2j-1,2\ell+1} - \mathrm{i}T^{2j,2\ell+1}) \end{aligned} \tag{5.4}$$

for $i, j = 1, 2, \ldots, \ell$ and $\varepsilon, \eta = \pm 1$. The matrices (5.2) and (5.3) obey the commutation relations

$$[H^j, H^k] = 0\,, \qquad [E^j_+, E^k_-] = \delta_{j,k}\, H^j\,, \qquad [H^j, E^k_\pm] = \pm\,(\alpha^{(k)})^j\, E^k_\pm \tag{5.5}$$

for $j, k = 1, 2, \ldots, \ell$, with structure constants

$$\begin{aligned} (\alpha^{(k)})^j &= \delta_{j,k} - \delta_{j,k+1} \qquad \text{for } k = 1, 2, \ldots, \ell-1\,, \\ (\alpha^{(\ell)})^j &= \begin{cases} \delta_{j,\ell-1} + \delta_{j,\ell} & \text{for } N = 2\ell\,, \\ \delta_{j,\ell} & \text{for } N = 2\ell+1\,. \end{cases} \end{aligned} \tag{5.6}$$

It is also straightforward to check that the matrices $E^j_\pm$ ($j = 1, 2, \ldots, \ell$) obey the Serre relations of so(N). It follows that (5.2) and (5.3) constitute a Cartan–Weyl basis for the defining matrix realization of so(N). The Cartan subalgebra is spanned by the H^j; the vectors $\alpha^{(k)}$ are the simple roots of so(N), and E^k_+ are the step operators corresponding to these simple roots. The step operators corresponding to positive roots are then $t^{i,j}_{+,-}$ and $t^{i,j}_{+,+}$ with $1 \le i < j \le \ell$, and the one corresponding to the highest root θ is $t^{1,2}_{+,+}$.

According to the explicit expressions (5.6) we are working with an orthonormal basis for the weight space of so(N). The relation with the Dynkin basis is as follows. For $N = 2\ell$, the components μ^i, $i = 1, 2, \ldots, \ell$, of a weight λ in the orthonormal basis are related to the weights λ^i of λ in the Dynkin basis by

$$\mu^i = \sum_{j=i}^{\ell-2} \lambda^j + \tfrac{1}{2}\,(\lambda^{\ell-1} + \lambda^\ell) \quad \text{for } i = 1, 2, \ldots, \ell-1\,, \qquad \mu^\ell = \tfrac{1}{2}\,(\lambda^{\ell-1} - \lambda^\ell)\,, \tag{5.7}$$

or conversely, $\lambda^i = \mu^i - \mu^{i+1}$ for $i = 1, 2, \ldots, \ell-2$ and $\lambda^{\ell-1} = \mu^{\ell-1} + \mu^\ell$, $\lambda^\ell = \mu^{\ell-1} - \mu^\ell$. For $N = 2\ell+1$, the analogous relations read

$$\mu^i = \sum_{j=i}^{\ell-1} \lambda^j + \tfrac{1}{2}\,\lambda^\ell \tag{5.8}$$

for all $i = 1, 2, \ldots, \ell$, respectively $\lambda^i = \mu^i - \mu^{i+1}$ for $i = 1, 2, \ldots, \ell-1$, $\lambda^\ell = 2\mu^\ell$. Thus in particular for the fundamental weights $\Lambda_{(j)}$ of so(N), defined by

$$(\alpha^{(j)}, \Lambda_{(k)}) = \begin{cases} \tfrac{1}{2}\,\delta_{k,\ell} & \text{for } j = \ell,\; N = 2\ell+1\,, \\ \delta_{j,k} & \text{else}\,, \end{cases} \tag{5.9}$$

the components in the orthonormal basis are

$$\Lambda_{(j)} = \begin{cases} (\underbrace{1,1,\dots,1}_{j \text{ times}},0,0,\dots,0) & \text{for } j=1,2,\dots,\ell-2 \text{ or } j=\ell-1,\ N=2\ell+1\,, \\ \frac12\,(1,1,\dots,1,1,-1) & \text{for } j=\ell-1,\ N=2\ell\,, \\ \frac12\,(1,1,\dots,1,1,1) & \text{for } j=\ell\,. \end{cases} \tag{5.10}$$

Finally we note that the invariant bilinear form on so(N) is

$$(T^{i,j}|T^{k,l}) = \tfrac12\,\mathrm{tr}\,(T^{i,j}T^{k,l}) = \delta_{i,k}\delta_{j,l} - \delta_{i,l}\delta_{j,k}\,. \tag{5.11}$$

In particular, we have $(H^i|H^j) = \delta_{i,j} = (E^i_+|E^j_-)$, $(E^i_\pm|E^j_\pm) = 0$.

6 The affine Lie algebra $\widehat{\mathrm{so}}(N)$

Given the fermion modes (2.14), one defines their normal-ordered bilinears

$$J^{i,j}_m := \mathrm{i}\sum_{q=1}^{2}\big[B^{i,j;q}_m - B^{j,i;q}_m\big]\,, \tag{6.1}$$

with

$$B^{i,j;q}_m := \tfrac12 \sum_{r\in\mathbb{Z}+1/2} {:}b^{i;q}_r b^{j;q}_{m-r}{:} \tag{6.2}$$

for $q=1,2$ and $i,j=1,2,\dots,N$. One checks by direct computation that

$$[J^{i,j}_m, b^{k;q}_r] = \mathrm{i}\,(\delta_{j,k}b^{i;q}_{r+m} - \delta_{i,k}b^{j;q}_{r+m})\,, \tag{6.3}$$

and

$$[J^{i,j}_m, J^{k,l}_n] = \mathrm{i}\,(\delta_{j,k}J^{i,l}_{m+n} + \delta_{i,l}J^{j,k}_{m+n} - \delta_{j,l}J^{i,k}_{m+n} - \delta_{i,k}J^{j,l}_{m+n}) + 2\,m\,\delta_{m,-n}\,(\delta_{i,k}\delta_{j,l} - \delta_{i,l}\delta_{j,k})\,. \tag{6.4}$$

According to (6.4) (compare also (5.11)), the $J^{i,j}_m$ with $i<j$ provide a basis for the affine Lie algebra $\widehat{\mathrm{so}}(N)$ at fixed value $k^\vee = 2$ of the level. That the level of $\widehat{\mathrm{so}}(N)$ has the value 2 is of course a consequence of the summation over two species of fermions in (6.1); for a single fermion one obtains analogously the Lie algebra $\widehat{\mathrm{so}}(N)$ at level 1. Also note that in the orthonormal basis the highest weights of integrable highest weight modules satisfy $\mu^0+\mu^1+\mu^2 = k^\vee$.

A Chevalley basis of the affine Lie algebra $\widehat{\mathrm{so}}(N)$ looks as follows. The Cartan subalgebra generators are

$$\mathcal{H}^j := J^{2j-1,2j}_0 \tag{6.5}$$

for $j=1,2,\dots,\ell$, and the step operators for the simple roots (respectively minus the simple roots) are $\mathcal{E}^j_\pm$ with $j=0,1,\dots,\ell$, given by

$$\begin{aligned} &\mathcal{E}^j_\pm = \pm J_0(t^{j,j+1}_{\pm,\mp}) \qquad \text{for } j=1,2,\dots,\ell-1\,, \\ &\mathcal{E}^0_\pm = \pm J_{\pm1}(t^{1,2}_{\mp,\mp})\,, \qquad \mathcal{E}^\ell_\pm = \begin{cases} \pm J_0(t^{\ell-1,\ell}_{\pm,\pm}) & \text{for } N=2\ell\,, \\ \pm J_0(t^{\ell}_{\pm}) & \text{for } N=2\ell+1\,, \end{cases} \end{aligned} \tag{6.6}$$

where

$$
\begin{aligned}
J_m(t^{i,j}_{\varepsilon,\eta}) &:= \tfrac{1}{2}\,(\varepsilon J_m^{2i,2j-1} + \eta J_m^{2i-1,2j}) + \tfrac{\mathrm{i}}{2}\,(J_m^{2i-1,2j-1} - \varepsilon\eta J_m^{2i,2j})\,, \\
J_m(t^{j}_{\varepsilon}) &:= -\frac{1}{\sqrt{2}}\,(\varepsilon J_m^{2j-1,2\ell+1} - \mathrm{i} J_m^{2j,2\ell+1})
\end{aligned}
\tag{6.7}
$$

for $i,j = 1,2,\ldots,\ell$ and $\varepsilon,\eta = \pm 1$.

At any integral level $k^\vee$ the affine Lie algebra $\widehat{\mathrm{so}}(N)$ has a finite number of irreducible highest weight modules $\mathcal{H}_\Lambda^{(k^\vee)}$; in the algebraic field theory description they correspond to the positive energy representations of the loop group $LSO(N)$ that are carried by the superselection sectors. For level one and level two these are listed in the tables 1 – 3. In these tables, Λ denotes the highest weight with respect to the horizontal subalgebra $\mathrm{so}(N)$, Δ the conformal weight, and $\mathcal{D}$ the statistical (or quantum) dimension. In the first column we provide a 'name' for the associated primary field of the relevant WZW theory; below we will use these names as labels for the irreducible highest weight modules, i.e. write $\mathcal{H}_\Lambda^{(2)} = \mathcal{H}_\circ^{(2)}$ for $\Lambda = 0$ etc., and for other quantities such as characters. (We find it convenient to use identical names for some of the fields at level one and at level two; when required to avoid ambiguities in the notation, we will always also specify the level.)

Table 1: Irreducible highest weight modules of $\widehat{\mathrm{so}}(N)$ at level 1 for $N = 2\ell$ (left) and for $N = 2\ell + 1$ (right).

field	Λ	Δ	$\mathcal{D}$
$\circ$	0	0	1
v	$\Lambda_{(1)}$	$\frac{1}{2}$	1
s	$\Lambda_{(\ell-1)}$	$\frac{N}{16}$	1
c	$\Lambda_{(\ell)}$	$\frac{N}{16}$	1

field	Λ	Δ	$\mathcal{D}$
$\circ$	0	0	1
v	$\Lambda_{(1)}$	$\frac{1}{2}$	1
σ	$\Lambda_{(\ell)}$	$\frac{N}{16}$	$\sqrt{2}$

In the tables we have separated the modules by a horizontal line into two classes. In the fermionic description, the modules in the first part are in the Neveu–Schwarz sector, while those in the second part are in the Ramond sector. As we only treat the Neveu–Schwarz sector of the fermions here, we will not deal with the second class of representations; we have included them in the tables only for completeness. Thus at level one we have $\mathcal{H}_{\mathrm{NS}} = \mathcal{H}_\circ^{(1)} \oplus \mathcal{H}_\mathrm{v}^{(1)}$, and hence at level two we can write

$$
\hat{\mathcal{H}}_{\mathrm{NS}} = (\mathcal{H}_\circ^{(1)} \otimes \mathcal{H}_\circ^{(1)}) \oplus (\mathcal{H}_\circ^{(1)} \otimes \mathcal{H}_\mathrm{v}^{(1)}) \oplus (\mathcal{H}_\mathrm{v}^{(1)} \otimes \mathcal{H}_\circ^{(1)}) \oplus (\mathcal{H}_\mathrm{v}^{(1)} \otimes \mathcal{H}_\mathrm{v}^{(1)})\,. \tag{6.8}
$$

The four summands in this decomposition can be characterized as the common eigenspaces with respect to the 'fermion flips' $U(\gamma_\pi\eta)$ and $U(\eta)$, namely those associated to the pairs $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$ of eigenvalues, respectively. By comparison with the action (4.6) and (4.7) of $\mathrm{O}(2)$ on the $\mathfrak{A}$ sectors, it follows that we can decompose the tensor products appearing in (6.8) as

$$
\begin{aligned}
&\mathcal{H}_\circ^{(1)} \otimes \mathcal{H}_\circ^{(1)} = \mathcal{H}_0 \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}_{[2n]}\,, \qquad \mathcal{H}_\mathrm{v}^{(1)} \otimes \mathcal{H}_\mathrm{v}^{(1)} = \mathcal{H}_J \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}_{[2n]}\,, \\
&\mathcal{H}_\circ^{(1)} \otimes \mathcal{H}_\mathrm{v}^{(1)} = \bigoplus_{n=0}^{\infty} \mathcal{H}_{[2n+1]} = \mathcal{H}_\mathrm{v}^{(1)} \otimes \mathcal{H}_\circ^{(1)}\,.
\end{aligned}
\tag{6.9}
$$

Table 2: Irreducible highest weight modules of $\widehat{\mathrm{so}}(N)$ at level 2 for $N = 2\ell$.

field	Λ	Δ	$\mathcal{D}$
$\circ$	0	0	1
v	$2\Lambda_{(1)}$	1	1
s	$2\Lambda_{(\ell-1)}$	$\frac{N}{8}$	1
c	$2\Lambda_{(\ell)}$	$\frac{N}{8}$	1
j	$\begin{cases} \Lambda_{(j)} \;\text{ for } j = 1, 2, \dots, \ell-2, \\ \Lambda_{(\ell-1)} + \Lambda_{(\ell)} \text{ for } j = \ell-1 \end{cases}$	$\frac{j(N-j)}{2N}$	2
σ	$\Lambda_{(\ell-1)}$	$\frac{N-1}{16}$	$\sqrt{N/2}$
τ	$\Lambda_{(\ell)}$	$\frac{N-1}{16}$	$\sqrt{N/2}$
σ'	$\Lambda_{(1)} + \Lambda_{(\ell-1)}$	$\frac{N+7}{16}$	$\sqrt{N/2}$
τ'	$\Lambda_{(1)} + \Lambda_{(\ell)}$	$\frac{N+7}{16}$	$\sqrt{N/2}$

Table 3: Irreducible highest weight modules of $\widehat{\mathrm{so}}(N)$ at level 2 for $N = 2\ell + 1$.

field	Λ	Δ	$\mathcal{D}$
$\circ$	0	0	1
v	$2\Lambda_{(1)}$	1	1
j	$\begin{cases} \Lambda_{(j)} \;\text{ for } j = 1, 2, \dots, \ell-1, \\ 2\Lambda_{(\ell)} \text{ for } j = \ell \end{cases}$	$\frac{j(N-j)}{2N}$	2
σ	$\Lambda_{(\ell-1)}$	$\frac{N-1}{16}$	$\sqrt{(N-1)/2}$
σ'	$\Lambda_{(1)} + \Lambda_{(\ell-1)}$	$\frac{N+7}{16}$	$\sqrt{(N-1)/2}$

The (Virasoro specialized) character of an irreducible highest weight module is the trace of q^{L_0} over the module, where L_0 is the zero mode of the energy-momentum tensor (see (3.5)) and where q is either regarded as a formal variable, or as $q = \exp(2\pi \mathrm{i}\tau)$ with τ in the upper complex half plane.[3] The characters of the modules in the Neveu–Schwarz sector at level one are

$$\chi_{\circ}^{(1)}(q) = \frac{(\varphi(-q^{1/2}))^N + (\varphi(q^{1/2}))^N}{2\,(\varphi(q))^N}\,, \qquad \chi_{\mathrm{v}}^{(1)}(q) = \frac{(\varphi(-q^{1/2}))^N - (\varphi(q^{1/2}))^N}{2\,(\varphi(q))^N}\,, \tag{6.10}$$

where

$$\varphi(q) := \prod_{n=1}^{\infty}(1-q^n) \tag{6.11}$$

is Euler's product function.

In section 9 we will employ the representation theory of the gauge group O(2), and in particular the decomposition (6.9), to obtain also simple formulæ for the characters of the level-two modules in the Neveu–Schwarz sector. As further input, we will need some information about the relevant coset conformal field theories.

7 $c = 1$ orbifolds

Via the coset construction [14], one associates to any embedding of untwisted affine Lie algebras that is induced by an embedding of their horizontal subalgebras another conformal field theory, called the coset theory. Here the relevant embedding is that of $\widehat{\mathrm{so}}(N)_2$ into $\widehat{\mathrm{so}}(N)_1 \oplus \widehat{\mathrm{so}}(N)_1$; the branching rules of this embedding are just the tensor product decompositions of $\widehat{\mathrm{so}}(N)_1$-modules.

The Virasoro algebra of the coset theory is easily obtained as the difference of the Sugawara constructions of the Virasoro algebras of the affine Lie algebras. In contrast, the determination of the field contents of the coset theory is in general a different task (see e.g. [15, 16]). But in the case of our interest, the coset theory has conformal central charge $c = 1$, and the classification of (unitary) $c = 1$ conformal field theories is well known. In fact, one finds (compare e.g. [12]) that it is a so-called rational $c = 1$ orbifold theory, which can be obtained from the $c = 1$ theory of a free boson compactified on a circle by restriction to the invariants with respect to a $\mathbb{Z}_2$-symmetry. These conformal field theory models have been investigated in [17]; for our purposes we need only the following information.

The rational $c = 1$ $\mathbb{Z}_2$-orbifolds are labelled by a non-negative integer M. The theory

[3] To obtain simpler transformation behavior with respect to modular transformations of the variable τ, one often defines the character with an additional factor of $q^{-c/24}$. For our purposes, this modification is not needed.

at a given value of M has $M+7$ sectors; they are listed in the following table.

field	Δ	$\mathcal{D}$
$\circ$	0	1
v	1	1
s, c	$\frac{M}{4}$	1
$j\ \in\{1,2,\ldots,M-1\}$	$\frac{j^2}{4M}$	2
$\sigma,\ \tau$	$\frac{1}{16}$	$\sqrt{M}$
$\sigma',\ \tau'$	$\frac{9}{16}$	$\sqrt{M}$

(7.1)

Here we have again separated the fields which correspond to the Neveu–Schwarz sector from the fields σ, τ, σ', τ' which correspond to the Ramond sector; the latter are known as 'twist fields' of the orbifold theory.

The characters of the fields in the Neveu–Schwarz sector are given by

$$\chi^{\mathrm{c};M}_{[j]}(q)=\frac{1}{\varphi(q)}\,\psi_{M,j}(q) \tag{7.2}$$

for $j=1,2,\ldots,M$, where it is understood that

$$\chi^{\mathrm{c};M}_{\mathrm{s}}(q)=\chi^{\mathrm{c};M}_{\mathrm{c}}(q)=\tfrac{1}{2}\,\chi^{\mathrm{c};M}_{[M]}\,, \tag{7.3}$$

and by

$$\chi^{\mathrm{c};M}_{\circ}(q)=\frac{1}{2\varphi(q)}\left[\psi_{M,0}(q)+\psi_{1,0}(-q)\right], \qquad \chi^{\mathrm{c};M}_{\mathrm{v}}(q)=\frac{1}{2\varphi(q)}\left[\psi_{M,0}(q)-\psi_{1,0}(-q)\right]. \tag{7.4}$$

Here the functions $\psi_{M,j}$ are the infinite sums

$$\psi_{M,j}(q):=\sum_{m\in\mathbb{Z}}q^{(j+2mM)^2/4M}\,. \tag{7.5}$$

One has [10, p. 240]

$$\psi_{1,0}(-q)=\sum_{m\in\mathbb{Z}}(-1)^m\,q^{m^2}=\frac{(\varphi(q))^2}{\varphi(q^2)}\,. \tag{7.6}$$

It follows in particular that

$$\chi^{\mathrm{c};M}_{\circ}(q)-\chi^{\mathrm{c};M}_{\mathrm{v}}(q)=\frac{\varphi(q)}{\varphi(q^2)}\,, \tag{7.7}$$

and

$$\chi^{\mathrm{c};M}_{\circ}(q)+\chi^{\mathrm{c};M}_{\mathrm{v}}(q)=\frac{\psi_{M,0}(q)}{\varphi(q)}\,. \tag{7.8}$$

Note that the spectrum of WZW theories for even and odd N, displayed in tables 1 – 3, is rather similar. However, to obtain the spectrum of the coset theory also the structure of the conjugacy classes of so(N)-modules play an important rôle, and these are

rather different for even and odd N.[4] As a consequence it depends on whether N is even or odd which $c = 1$ orbifold one obtains as the coset theory. Namely, for $N = 2\ell$ one finds $M = N/2 = \ell$, while $M = 2N$ for $N = 2\ell + 1$.

The decomposition of the products of level one characters looks as follows. For $N = 2\ell$ we have

$$\begin{aligned}
[\chi^{(1)}_{\circ}]^2 &= \chi^{\mathrm{c};\ell}_{\circ}\,\chi^{(2)}_{\circ} + \chi^{\mathrm{c};\ell}_{\mathrm{v}}\,\chi^{(2)}_{\mathrm{v}} + \sum_{\substack{2\le j\le\ell \\ j\ \mathrm{even}}} \chi^{\mathrm{c};\ell}_{[j]}\,\chi^{(2)}_{[j]}\,, \\
[\chi^{(1)}_{\mathrm{v}}]^2 &= \chi^{\mathrm{c};\ell}_{\circ}\,\chi^{(2)}_{\mathrm{v}} + \chi^{\mathrm{c};\ell}_{\mathrm{v}}\,\chi^{(2)}_{\circ} + \sum_{\substack{2\le j\le\ell \\ j\ \mathrm{even}}} \chi^{\mathrm{c};\ell}_{[j]}\,\chi^{(2)}_{[j]}\,, \\
\chi^{(1)}_{\circ}\,\chi^{(1)}_{\mathrm{v}} &= \sum_{\substack{1\le j\le\ell \\ j\ \mathrm{odd}}} \chi^{\mathrm{c};\ell}_{[j]}\,\chi^{(2)}_{[j]}\,,
\end{aligned} \tag{7.9}$$

where it is understood that

$$\chi^{(2)}_{[\ell]}(q) \equiv \chi^{(2)}_{\mathrm{s}}(q) + \chi^{(2)}_{\mathrm{c}}(q)\,. \tag{7.10}$$

For $N = 2\ell + 1$, the tensor product decomposition reads instead

$$\begin{aligned}
[\chi^{(1)}_{\circ}]^2 &= \chi^{\mathrm{c};2N}_{\circ}\,\chi^{(2)}_{\circ} + \chi^{\mathrm{c};2N}_{\mathrm{v}}\,\chi^{(2)}_{\mathrm{v}} + \sum_{\substack{2\le j\le\ell \\ j\ \mathrm{even}}} \chi^{\mathrm{c};2N}_{[2j]}\,\chi^{(2)}_{[j]} + \sum_{\substack{1\le j\le\ell \\ j\ \mathrm{odd}}} \chi^{\mathrm{c};2N}_{[2N-2j]}\,\chi^{(2)}_{[j]}\,, \\
[\chi^{(1)}_{\mathrm{v}}]^2 &= \chi^{\mathrm{c};2N}_{\circ}\,\chi^{(2)}_{\mathrm{v}} + \chi^{\mathrm{c};2N}_{\mathrm{v}}\,\chi^{(2)}_{\circ} + \sum_{\substack{2\le j\le\ell \\ j\ \mathrm{even}}} \chi^{\mathrm{c};2N}_{[2j]}\,\chi^{(2)}_{[j]} + \sum_{\substack{1\le j\le\ell \\ j\ \mathrm{odd}}} \chi^{\mathrm{c};2N}_{[2N-2j]}\,\chi^{(2)}_{[j]}\,, \\
\chi^{(1)}_{\circ}\,\chi^{(1)}_{\mathrm{v}} &= \chi^{\mathrm{c};2N}_{[2N]}\left[\chi^{(2)}_{\circ} + \chi^{(2)}_{\mathrm{v}}\right] + \sum_{\substack{2\le j\le\ell \\ j\ \mathrm{even}}} \chi^{\mathrm{c};2N}_{[2N-2j]}\,\chi^{(2)}_{[j]} + \sum_{\substack{1\le j\le\ell \\ j\ \mathrm{odd}}} \chi^{\mathrm{c};2N}_{[2j]}\,\chi^{(2)}_{[j]}\,.
\end{aligned} \tag{7.11}$$

It is worth noting that these formulæ can be proven without too much effort, whereas in general it is a difficult task to write down such tensor product decompositions. Tools which are always available are the matching of conformal dimensions modulo integers as well as conjugacy class selection rules, which imply [15] so-called field identifications. In the present case, we can e.g. use the fact that the sum of conformal weights $\Delta^{(2)}_j = j(N-j)/2N$ and $\Delta^{\mathrm{c};M}_k = k^2/4M$ is (for generic N) a half-integer only if $k = j\sqrt{2M/N}$ or $k = (N-j)\sqrt{2M/N}$. Also, there is a conjugacy class selection rule which implies that the tensor product of modules in the Neveu–Schwarz sector yields only modules which are again in the Neveu–Schwarz sector, and the corresponding field identification tells us e.g. that the branching function $b^{\mathrm{c};M}_{\mathrm{v,v;v}}(q)$ coincides with $b^{\mathrm{c};M}_{\circ,\circ;\circ}(q) = \chi^{\mathrm{c};M}_{\circ}(q)$.

As it turns out, we are even in the fortunate situation that together with the known classification of unitary $c = 1$ conformal field theories, these informations already determine the tensor product decompositions almost completely. In particular, the value of M of the $c = 1$ orbifold is determined uniquely, and one can prove that there aren't any further field identifications besides the ones implied by conjugacy class selection rules. The remaining ambiguities can be resolved by checking various consistency relations which follow from the arguments that we will give in section 9 below. Another possibility to

[4] Also, for odd N in the Ramond sector an additional complication arises, namely a so-called fixed point resolution is required [12, 16].

deduce (7.9) and (7.11) is to employ the conformal embedding of $\widehat{\mathrm{so}}(N)_2$ into $\widehat{\mathrm{u}}(N)$ at level one [12], which corresponds to regarding the real fermions as real and imaginary parts of complex-valued fermions.

8 Highest weight vectors of $\widehat{\mathrm{so}}(N)_2$

8.1 Definition of the vectors

A highest weight vector $|\Phi_\Lambda\rangle$ of $\widehat{\mathrm{so}}(N)_2$ with highest weight Λ is characterized by the following properties. First, it is annihilated by the step operators associated to the horizontal positive roots, i.e. for $1 \le i < j \le \ell$ and $\varepsilon = \pm 1$ one has

$$J_0(t^{i,j}_{+,\varepsilon})\,|\Phi_\Lambda\rangle = 0\,, \qquad \text{and also} \qquad J_0(t^k_+)\,|\Phi_\Lambda\rangle = 0 \;\text{ for } N = 2\ell+1\,; \tag{8.1}$$

second, it is also annihilated by the step operators with positive grade, i.e. for $m > 0$, $i,j = 1,2,\ldots,\ell$ and $\varepsilon,\eta = \pm 1$ it satisfies

$$J_m(t^{i,j}_{\varepsilon,\eta})\,|\Phi_\Lambda\rangle = 0\,, \qquad \text{and also} \qquad J_m(t^k_\varepsilon)\,|\Phi_\Lambda\rangle = 0 \;\text{ for } N = 2\ell+1\,; \tag{8.2}$$

and third, $|\Phi_\Lambda\rangle$ is an eigenvector of the Cartan subalgebra,

$$\mathcal{H}^k\,|\Phi_\Lambda\rangle = \Lambda^k\,|\Phi_\Lambda\rangle \tag{8.3}$$

for $k = 1,2,\ldots,\ell$.

We will exploit the decomposition of $\hat{\mathcal{H}}_{\mathrm{NS}}$ into irreducible $\mathfrak{A}$ sectors to identify the highest weight vectors of $\widehat{\mathrm{so}}(N)_2$. Indeed, in each sector $\mathcal{H}_0$, $\mathcal{H}_J$ and $\mathcal{H}^\pm_{[m]}$ we find distinguished states which are highest weight vectors for both $\widehat{\mathrm{so}}(N)_2$ and the coset Virasoro algebra. The construction works as follows. The vector $|\Omega\rangle \equiv |\hat{\Omega}_{\mathrm{NS}}\rangle$ is a highest weight state of $\mathfrak{g} = \widehat{\mathrm{so}}(N)_2$ with highest weight zero. To describe more highest weight vectors, it is helpful to introduce some notation. We define

$$x^{j,\pm}_r := \frac{1}{\sqrt{2}}\,(c^{j,+}_r \pm \mathrm{i}\bar{c}^{j,+}_r)\,, \qquad \bar{x}^{j,\pm}_r := \frac{1}{\sqrt{2}}\,(c^{j,-}_r \pm \mathrm{i}\bar{c}^{j,-}_r)\,, \tag{8.4}$$

for $j = 1,2,\ldots,\ell$, where

$$c^{j,\pm}_r := \frac{1}{\sqrt{2}}\,(b^{2j;1}_r \pm \mathrm{i}b^{2j-1;1}_r)\,, \qquad \bar{c}^{j,\pm}_r := \frac{1}{\sqrt{2}}\,(b^{2j;2}_r \pm \mathrm{i}b^{2j-1;2}_r)\,, \tag{8.5}$$

and also, for $N = 2\ell+1$,

$$\bar{x}^{\ell+1,\pm}_r := \frac{1}{\sqrt{2}}\,(b^{2\ell+1;1}_r \pm \mathrm{i}b^{2\ell+1;2}_r)\,. \tag{8.6}$$

Further, we set

$$\begin{aligned} X^{j,\pm}_r &:= x^{j,\pm}_r\, x^{j-1,\pm}_r \cdots x^{1,\pm}_r \qquad &&\text{for } j = 1,2,\ldots,\ell\,,\\ \bar{X}^{j,\pm}_r &:= \bar{x}^{j+1,\pm}_r\, \bar{x}^{j+2,\pm}_r \cdots \bar{x}^{\ell,\pm}_r \qquad &&\text{for } j = 0,1,\ldots,\ell-1\,, \end{aligned} \tag{8.7}$$

and $\bar{X}^{\ell,\pm}_r := \mathbf{1}$.

For any $n = 0,1,2,\ldots$ we can now define the following vectors: We set

$$|\Omega^{n,\pm}_{[j]}\rangle := X^{j,\pm}_{-n-1/2}\,|\Omega^{n,\pm}_\circ\rangle\,, \qquad \text{for } j = 1,2,\ldots,\ell\,, \tag{8.8}$$

as well as

$$|\overline{\Omega}^{n,\pm}_{[j]}\rangle := \begin{cases} \bar{X}^{j,\pm}_{-n-1/2} X^{\ell,\pm}_{-n-1/2}\,|\Omega^{n,\pm}_{\circ}\rangle & \text{for } N = 2\ell\,,\ j = 1,2,\ldots,\ell-1\,, \\ \bar{X}^{j,\pm}_{-n-1/2} \bar{x}^{\ell+1,\pm}_{-n-1/2} X^{\ell,\pm}_{-n-1/2}\,|\Omega^{n,\pm}_{\circ}\rangle & \text{for } N = 2\ell+1\,,\ j = 1,2,\ldots,\ell\,. \end{cases} \tag{8.9}$$

Here we defined recursively

$$|\Omega^{n+1,\pm}_{\circ}\rangle := \begin{cases} \bar{X}^{0,\pm}_{-n-1/2} X^{\ell,\pm}_{-n-1/2}\,|\Omega^{n,\pm}_{\circ}\rangle & \text{for } N = 2\ell\,, \\ \bar{X}^{0,\pm}_{-n-1/2} \bar{x}^{\ell+1,\pm}_{-n-1/2} X^{\ell,\pm}_{-n-1/2}\,|\Omega^{n,\pm}_{\circ}\rangle & \text{for } N = 2\ell+1\,, \end{cases} \tag{8.10}$$

with

$$|\Omega^{0,\pm}_{\circ}\rangle := |\Omega\rangle \qquad . \tag{8.11}$$

Further, we set

$$|\Omega_{\mathrm{v}}\rangle \equiv \pm|\Omega^{0,\pm}_{\mathrm{v}}\rangle := x^{1,+}_{-1/2} x^{1,-}_{-1/2}\,|\Omega\rangle\,, \quad |\Omega^{n,\pm}_{\mathrm{v}}\rangle := x^{1,\pm}_{-n-1/2} x^{1,\mp}_{n-1/2}\,|\Omega^{n,\pm}_{\circ}\rangle\,,\ \ n = 1,2,\ldots\,, \tag{8.12}$$

and, for $N = 2\ell$,

$$|\Omega^{n,\pm}_{\mathrm{s}}\rangle := |\Omega^{n,\pm}_{[\ell]}\rangle\,, \qquad |\Omega^{n,\pm}_{\mathrm{c}}\rangle := \bar{x}^{\ell,\pm}_{-n-1/2} \bar{x}^{\ell,\mp}_{n+1/2}\,|\Omega^{n,\pm}_{\mathrm{s}}\rangle\,. \tag{8.13}$$

8.2 O(2) transformation properties

The vacuum $|\Omega\rangle$ is O(2)-invariant. We then deduce (compare the appendix, (A.9)–(A.13)) the following transformations for the vectors (8.8)–(8.13). For all $n = 0,1,2,\ldots$ we have

$$U(\gamma_t)\,|\Omega^{n,\pm}_{[j]}\rangle = \mathrm{e}^{\pm\mathrm{i}(nN+j)t}\,|\Omega^{n,\pm}_{[j]}\rangle\,, \qquad U(\eta)\,|\Omega^{n,\pm}_{[j]}\rangle = |\Omega^{n,\mp}_{[j]}\rangle \tag{8.14}$$

for $j = 1,2,\ldots,\ell$, and

$$U(\gamma_t)\,|\overline{\Omega}^{n,\pm}_{[j]}\rangle = \mathrm{e}^{\pm\mathrm{i}((n+1)N-j)t}\,|\overline{\Omega}^{n,\pm}_{[j]}\rangle\,, \qquad U(\eta)\,|\overline{\Omega}^{n,\pm}_{[j]}\rangle = |\overline{\Omega}^{n,\mp}_{[j]}\rangle \tag{8.15}$$

for $j = 1,2,\ldots,\ell$. Also

$$\begin{aligned} U(\gamma_t)\,|\Omega^{n,\pm}_{\circ}\rangle &= \mathrm{e}^{\pm\mathrm{i}nNt}\,|\Omega^{n,\pm}_{\circ}\rangle\,, & U(\eta)\,|\Omega^{n,\pm}_{\circ}\rangle &= |\Omega^{n,\mp}_{\circ}\rangle\,, \\ U(\gamma_t)\,|\Omega^{n,\pm}_{\mathrm{v}}\rangle &= \mathrm{e}^{\pm\mathrm{i}nNt}\,|\Omega^{n,\pm}_{\mathrm{v}}\rangle\,, & U(\eta)\,|\Omega^{n,\pm}_{\mathrm{v}}\rangle &= |\Omega^{n,\mp}_{\mathrm{v}}\rangle\,, \\ U(\gamma_t)\,|\Omega^{n,\pm}_{\mathrm{s}}\rangle &= \mathrm{e}^{\pm\mathrm{i}(nN+\ell)t}\,|\Omega^{n,\pm}_{\mathrm{s}}\rangle\,, & U(\eta)\,|\Omega^{n,\pm}_{\mathrm{s}}\rangle &= |\Omega^{n,\mp}_{\mathrm{s}}\rangle\,, \\ U(\gamma_t)\,|\Omega^{n,\pm}_{\mathrm{c}}\rangle &= \mathrm{e}^{\pm\mathrm{i}(nN+\ell)t}\,|\Omega^{n,\pm}_{\mathrm{c}}\rangle\,, & U(\eta)\,|\Omega^{n,\pm}_{\mathrm{c}}\rangle &= |\Omega^{n,\mp}_{\mathrm{c}}\rangle\,. \end{aligned} \tag{8.16}$$

We remark that the highest weight states $|\Omega^{n,\pm}_{\mathrm{v}}\rangle$ and $|\Omega^{n,\pm}_{\circ}\rangle$, $n = 1,2,\ldots$, and for even N also $|\Omega^{n,\pm}_{\mathrm{c}}\rangle$ and $|\Omega^{n,\pm}_{\mathrm{s}}\rangle$, $n = 0,1,2,\ldots$, are connected by O(2)-invariant fermion bilinears, i.e. by elements of the intermediate algebra $\mathfrak{A}$. Explicitly, we have

$$|\Omega^{n,\pm}_{\mathrm{v}}\rangle = a^n_{\mathrm{v}}\,|\Omega^{n,\pm}_{\circ}\rangle, \qquad a^n_{\mathrm{v}} = -(x^{1,-}_{n-1/2} x^{1,+}_{-n-1/2} + x^{1,+}_{n-1/2} x^{1,-}_{-n-1/2})\,, \tag{8.17}$$

for $n = 1,2,\ldots$, and

$$|\Omega^{n,\pm}_{\mathrm{c}}\rangle = a^n_{\mathrm{c}}\,|\Omega^{n,\pm}_{\mathrm{s}}\rangle, \qquad a^n_{\mathrm{c}} = -(\bar{x}^{\ell,-}_{n+1/2} \bar{x}^{\ell,+}_{-n-1/2} + \bar{x}^{\ell,+}_{n+1/2} \bar{x}^{\ell,-}_{-n-1/2}) \tag{8.18}$$

for $n = 0,1,2,\ldots$.

8.3 The highest $\widehat{\mathrm{so}}(N)_2$ weights

The states defined above are eigenvectors of all Cartan subalgebra generators $\mathcal{H}^k$ ($k = 1,2,\dots,\ell$) and of the central generator K; the level $k^\vee$ is equal to 2, and the weights do not depend on the label n. More precisely, from the commutation relations (A.1) and (A.2) it follows rather directly that

$$\begin{aligned} \mathcal{H}^k\,|\Omega^{n,\pm}_{[j]}\rangle &= (\Lambda_{[j]})^k\,|\Omega^{n,\pm}_{[j]}\rangle \quad \text{for } j=1,2,\dots,\ell\,,\\ \mathcal{H}^k\,|\overline{\Omega}^{n,\pm}_{[j]}\rangle &= (\Lambda_{[j]})^k\,|\overline{\Omega}^{n,\pm}_{[j]}\rangle \quad \text{for } j=1,2,\dots,\ell-1 \end{aligned} \tag{8.19}$$

and

$$\begin{aligned} \mathcal{H}^k\,|\Omega^{n,\pm}_{\circ}\rangle &= (\Lambda_{\circ})^k\,|\Omega^{n,\pm}_{\circ}\rangle\,, \qquad & \mathcal{H}^k\,|\Omega^{n,\pm}_{\mathrm{v}}\rangle &= (\Lambda_{\mathrm{v}})^k\,|\Omega^{n,\pm}_{\mathrm{v}}\rangle\,,\\ \mathcal{H}^k\,|\Omega^{n,\pm}_{\mathrm{s}}\rangle &= (\Lambda_{\mathrm{s}})^k\,|\Omega^{n,\pm}_{\mathrm{s}}\rangle\,, \qquad & \mathcal{H}^k\,|\Omega^{n,\pm}_{\mathrm{c}}\rangle &= (\Lambda_{\mathrm{c}})^k\,|\Omega^{n,\pm}_{\mathrm{c}}\rangle\,. \end{aligned} \tag{8.20}$$

The weights $\Lambda_{[j]}$ appearing here are those listed in table 2 and 3, i.e. we have

$$\Lambda_{[j]} = \begin{cases} \Lambda_{(j)} & \text{for } j=1,2,\dots,\ell-2 \ \text{ or } \ j=\ell-1,\ N=2\ell+1\,,\\ \Lambda_{(\ell-1)}+\Lambda_{(\ell)} & \text{for } j=\ell-1,\ N=2\ell\,,\\ 2\Lambda_{(\ell)} & \text{for } j=\ell,\ N=2\ell+1\,, \end{cases} \tag{8.21}$$

with the fundamental weights $\Lambda_{(i)}$ as defined in (5.10), while $\Lambda_\circ = 0$, $\Lambda_{\mathrm{v}} = 2\Lambda_{(1)}$, and, for $N=2\ell$, $\Lambda_{\mathrm{s}} = 2\Lambda_{(\ell)}$, $\Lambda_{\mathrm{c}} = 2\Lambda_{(\ell-1)}$.

8.4 The highest weight property

Having obtained (8.19), for proving that the states (8.8) – (8.13) are highest weight vectors with respect to $\widehat{\mathrm{so}}(N)_2$ it is now sufficient to show that they are annihilated by $\mathcal{E}^j_+$ for $j=0,1,\dots,\ell$. This can easily be checked by inserting the results (A.6) – (A.8) for the commutators between the step operators $\mathcal{E}^j_+$ and the operators $X^{k,\pm}_r$, $\bar X^{k,\pm}_r$ into the definitions of these states. The least trivial case occurs for $\mathcal{E}^0_+$, where one employs the first of the identities (A.8); one then has to commute $\bar x^{1,\pm}_{1/2}$ and $\bar x^{2,\pm}_{1/2}$, to the right and use $\bar x^{1,\pm}_{1/2}|\Omega\rangle = 0 = \bar x^{2,\pm}_{1/2}|\Omega\rangle$ when $n=0$, while for $n>0$ one also must employ the second identity in (A.8).

Thus all the states (8.8) – (8.13) are highest weight states of $\mathfrak{g} = \widehat{\mathrm{so}}(N)_2$. We claim further that they are highest weight vectors with respect to the coset Virasoro algebra, too. This follows directly from the fact that L^{NS}_m with $m>0$ annihilates these states, which is a consequence of

$$[L^{\mathrm{NS}}_m, x^{j,\pm}_r] = -(r+\tfrac{m}{2})\,x^{j,\pm}_{r+m}\,, \qquad [L^{\mathrm{NS}}_m, \bar x^{j,\pm}_r] = -(r+\tfrac{m}{2})\,\bar x^{j,\pm}_{r+m}\,. \tag{8.22}$$

Since the affine Lie algebra $\widehat{\mathrm{so}}(N)_2$ and the coset Virasoro algebra commute, it follows immediately that further highest weight vectors of $\widehat{\mathrm{so}}(N)_2$ are obtained when acting with the creation operators of the coset Virasoro algebra on the vectors (8.8) – (8.13). For example, applying the coset Virasoro operator L^{c}_{-1} to the highest weight vector $|\Omega^+_{[1]}\rangle$ we get the highest weight vector (computed for the case $N=2\ell$)

$$L^{\mathrm{c}}_{-1}|\Omega^+_{[1]}\rangle = \frac{1}{N}\Big[x^{1,+}_{-3/2}|\Omega\rangle + \sum_{k=1}^{\ell}(\bar x^{k,+}_{-1/2}x^{k,-}_{-1/2} - \bar x^{k,-}_{-1/2}x^{k,+}_{-1/2})|\Omega^+_{[1]}\rangle\Big] \tag{8.23}$$

of $\widehat{\mathrm{so}}(N)_2$. However, it follows from the results in section 9 below that, except for a few special cases, the vectors (8.8) – (8.13) exhaust the set of simultaneous highest weight states of $\widehat{\mathrm{so}}(N)_2$ and the coset Virasoro algebra.

Also note that by construction the tensor product module, and hence each of its submodules, is unitary. Thus in particular the highest weight modules that are obtained by acting with arbitrary polynomials in the lowering operators $\mathcal{E}^i_-$ on the highest weight vectors are unitary, and hence are fully reducible.

8.5 Conformal weights

The action of the zero mode of the free fermion Virasoro algebra (3.5) on the fermion modes $x_r^{i,\pm}$ reads

$$[L_0^{\mathrm{NS}}, x_r^{i,\pm}] = -r\, x_r^{i,\pm}\,, \qquad [L_0^{\mathrm{NS}}, \bar{x}_r^{i,\pm}] = -r\, \bar{x}_r^{i,\pm}\,. \tag{8.24}$$

From these relations we deduce that

$$L_0^{\mathrm{NS}}\, |\Omega^{n,\pm}_{[j]}\rangle = \Delta^{\mathrm{NS}}_{n;j}\, |\Omega^{n,\pm}_{[j]}\rangle\,, \tag{8.25}$$

with conformal weights

$$\Delta^{\mathrm{NS}}_{n;j} = [\tfrac12 + \tfrac32 + \ldots + \big(n-\tfrac12\big)]N + \big(n+\tfrac12\big)j = \frac{n^2N}{2} + \big(n+\tfrac12\big)j \tag{8.26}$$

for $j = 1, 2, \ldots, \ell$. Similarly,

$$L_0^{\mathrm{NS}}\, |\overline{\Omega}^{n,\pm}_{[j]}\rangle = \bar{\Delta}^{\mathrm{NS}}_{n;j}\, |\overline{\Omega}^{n,\pm}_{[j]}\rangle\,, \qquad \bar{\Delta}^{\mathrm{NS}}_{n;j} = \frac{(n{+}1)^2N}{2} - \big(n+\tfrac12\big)j\,, \tag{8.27}$$

for $j = 1, 2, \ldots, \ell$. Also, for the sectors labelled by $\circ$, v, s and c we find

$$\Delta^{\mathrm{NS}}_{n;\circ} = \frac{n^2N}{2}\,, \qquad \Delta^{\mathrm{NS}}_{n;\mathrm{v}} = \frac{n^2N}{2} + 1\,, \qquad \Delta^{\mathrm{NS}}_{n;\mathrm{s}} = \Delta^{\mathrm{NS}}_{n;\mathrm{c}} = \Delta^{\mathrm{NS}}_{n;\ell}\,. \tag{8.28}$$

Furthermore, the conformal weights of the vectors (8.8) – (8.13) with respect to the Virasoro algebra of the level-two WZW theory follow immediately from the so(N)-weights Λ by the Sugawara formula for the Virasoro generator L_0. This yields the conformal weights that were already listed in the tables 2 and 3. Comparing these conformal dimensions with the ones obtained above, we arrive at the result

$$\begin{aligned} \Delta^{\mathrm{c}}_{n;j} &= \frac{1}{2N}(nN+j)^2\,, & j &= 1, 2, \ldots, \ell\,, \\ \bar{\Delta}^{\mathrm{c}}_{n;j} &= \frac{1}{2N}((n+1)N-j)^2\,, & j &= 1, 2, \ldots, \ell\,, \end{aligned} \tag{8.29}$$

and

$$\Delta^{\mathrm{c}}_{n;\circ} = \Delta^{\mathrm{c}}_{n;\mathrm{v}} = \frac{n^2N}{2}\,, \qquad \Delta^{\mathrm{c}}_{n;\mathrm{s}} = \Delta^{\mathrm{c}}_{n;\mathrm{c}} = \Delta^{\mathrm{c}}_{n;\ell} \tag{8.30}$$

for the eigenvalues of the coset Virasoro generator $L_0^{\mathrm{c}} = L_0^{\mathrm{NS}} - L_0$.

9 Characters of the modules in the Neveu–Schwarz sector

Because of the inclusion $\mathfrak{A}_{\rm WZW} \subset \mathfrak{A}$, the irreducible sectors of the gauge invariant fermion algebra $\mathfrak{A}$ constitute modules of the observable algebra $\mathfrak{A}_{\rm WZW}$ of the WZW theory, which however are typically reducible. To determine the decomposition of the irreducible modules of the intermediate algebra $\mathfrak{A}$ into irreducible modules of $\mathfrak{A}_{\rm WZW}$ we analyze their characters and combine the result with the knowledge about the characters of the coset theory.

9.1 Characters for the sectors of $\mathfrak{A}$

The characters of submodules of the space $\hat{\mathcal{H}}_{\rm NS}$, i.e. the trace of q^{L_0} over the modules, can be obtained as follows. Let P_0, P_J and $P^{\pm}_{[m]}$ denote the projections onto $\mathcal{H}_0, \mathcal{H}_J$ and $\mathcal{H}^{\pm}_{[m]}$ for $m\in\mathbb{N}$, respectively. Then the representation matrices $U(\gamma_t)$ and $U(\eta\gamma_t)$ of O(2) decompose into projectors as

$$U(\gamma_t) = P_0 + P_J + \sum_{m=1}^{\infty} \big[{\rm e}^{{\rm i}mt} P^{+}_{[m]} + {\rm e}^{-{\rm i}mt} P^{-}_{[m]} \big] \tag{9.1}$$

and

$$U(\eta\gamma_t) = P_0 - P_J + \sum_{m=1}^{\infty} \big[{\rm e}^{{\rm i}mt} U(\eta) P^{+}_{[m]} + {\rm e}^{-{\rm i}mt} U(\eta) P^{-}_{[m]} \big] \,. \tag{9.2}$$

It follows in particular that the projectors can be written as

$$\begin{aligned} &P_0 = \frac{1}{4\pi} \int_0^{2\pi} {\rm d}t \, \big[U(\gamma_t) + U(\eta\gamma_t) \big] \,, \qquad P_J = \frac{1}{4\pi} \int_0^{2\pi} {\rm d}t \, \big[U(\gamma_t) - U(\eta\gamma_t) \big] \,, \\ &P^{\pm}_{[m]} = \frac{1}{2\pi} \int_0^{2\pi} {\rm d}t \, {\rm e}^{\mp {\rm i}mt} \, U(\gamma_t) \qquad \text{for } m\in\mathbb{N} \,. \end{aligned} \tag{9.3}$$

For the irreducible $\mathfrak{A}$-sectors in $\hat{\mathcal{H}}_{\rm NS}$, the results (8.14) – (8.18) together with the action of $L_0^{\rm NS}$ (compare (8.24)) imply the following. First,

$$\begin{aligned} \chi_0^{\rm NS}(q) \; &\equiv {\rm tr}_{\hat{\mathcal{H}}_{\rm NS}} P_0 \, q^{L_0^{\rm (NS)}} \\ &= \frac{1}{4\pi} \int_0^{2\pi} {\rm d}t \, \big[\prod_{m=0}^{\infty} (1 + {\rm e}^{{\rm i}t} q^{m+1/2})^N (1 + {\rm e}^{-{\rm i}t} q^{m+1/2})^N + \prod_{m=0}^{\infty} (1 - q^{2m+1})^N \big] \,. \end{aligned} \tag{9.4}$$

This can be rewritten as

$$\chi_0^{\rm NS}(q) = \frac{1}{4\pi} \int_0^{2\pi} {\rm d}t \, \big[\frac{\xi(q; -{\rm e}^{{\rm i}t} q^{1/2})}{\varphi(q)} \big]^N + \frac{1}{2} \big[\frac{\varphi(q)}{\varphi(q^2)} \big]^N \,, \tag{9.5}$$

where φ is Euler's product function (6.11) and

$$\xi(q;z) := \prod_{n=1}^{\infty} \big((1-q^n)(1-q^n z^{-1})(1-q^{n-1} z) \big) \,. \tag{9.6}$$

Using also the identity $\xi(q;z) = \sum_{n\in\mathbb{Z}}(-1)^n\, q^{n(n-1)/2} z^n$ [10, p. 240], we finally arrive at

$$\chi_0^{\rm NS}(q) = \frac{\Theta_{N,0}(q)}{2\,(\varphi(q))^N} + \frac{(\varphi(q))^N}{2\,(\varphi(q^2))^N}\,, \tag{9.7}$$

where we introduced the functions

$$\Theta_{N,m}(q) := \sum_{\substack{m_1,m_2,\ldots,m_N\in\mathbb{Z}\\ m_1+m_2+\ldots+m_N=m}} q^{(m_1^2+m_2^2+\ldots+m_N^2)/2} \;\equiv\; \sum_{\substack{\mathbf{m}\in\mathbb{Z}^N\\ \sum m_i=m}} q^{\mathbf{m}^2/2} \tag{9.8}$$

for $m\in\mathbb{Z}$.

Analogously, we find

$$\chi_J^{\rm NS}(q) \equiv {\rm tr}_{\hat{\mathcal{H}}_{\rm NS}} P_J\, q^{L_0^{\rm (NS)}} = \frac{\Theta_{N,0}(q)}{2\,(\varphi(q))^N} - \frac{(\varphi(q))^N}{2\,(\varphi(q^2))^N} \tag{9.9}$$

and

$$\chi_{[m]}^{\rm NS}(q) \equiv {\rm tr}_{\hat{\mathcal{H}}_{\rm NS}} P_{[m]}^{\pm}\, q^{L_0^{\rm (NS)}} = \frac{1}{2\pi}\int_0^{2\pi} {\rm d}t\, {\rm e}^{\mp{\rm i}mt} \Big[\frac{\xi(q;-{\rm e}^{{\rm i}t}q^{1/2})}{\varphi(q)}\Big]^N = \frac{\Theta_{N,m}(q)}{(\varphi(q))^N} \tag{9.10}$$

for $m\in\mathbb{N}$. (Note that the latter result does not depend on whether $P_{[m]}^+$ or $P_{[m]}^-$ is used, since $\Theta_{N,m}(q) = \Theta_{N,-m}(q)$.)

Expressing the integer m either as $m = nN+j$ or as $m = (n+1)N-j$ with $1 \le j \le \ell$, by shifting the summation indices we obtain the relation $\Theta_{N,nN+j}(q) = q^{nj+n^2N/2}\,\Theta_{N,j}(q)$. Hence we have

$$\chi_{[nN+j]}^{\rm NS}(q) = q^{nj+n^2N/2}\,\chi_{[j]}^{\rm NS}(q)\,; \tag{9.11}$$

in the same manner we obtain $\chi_{[(n+1)N-j]}^{\rm NS}(q) = q^{n(N-j)+n^2N/2}\,\chi_{[N-j]}^{\rm NS}(q)$, or alternatively

$$\chi_{[(n+1)N-j]}^{\rm NS}(q) = q^{-(n+1)j+(n+1)^2N/2}\,\chi_{[j]}^{\rm NS}(q)\,. \tag{9.12}$$

For $j=0$ we have instead

$$\chi_{[nN]}^{\rm NS}(q) = q^{n^2N/2}\,[\chi_0^{\rm NS}(q) + \chi_J^{\rm NS}(q)] \tag{9.13}$$

for all $n>0$.

9.2 $\widehat{\rm so}(N)_2$ characters for even N

When we use the information about the highest weight vectors with respect to the affine Lie algebra $\widehat{\rm so}(N)$ at level 2 that we obtained above, we can derive the characters of the irreducible highest weight modules of $\widehat{\rm so}(N)_2$ by comparing the decomposition (6.9) with the decompositions (7.9) and (7.11). We first consider the case $N=2\ell$.

By comparison of (6.9) with (7.9) we find

$$\begin{aligned}\chi_{[j]}^{{\rm c};\ell}(q)\,\chi_{[j]}^{(2)}(q) &= \chi_{[j]}^{\rm NS}(q) + \chi_{[N-j]}^{\rm NS}(q) + \chi_{[N+j]}^{\rm NS}(q) + \chi_{[2N-j]}^{\rm NS}(q) + \ldots\\ &\equiv \sum_{n=0}^{\infty}[\chi_{[nN+j]}^{\rm NS}(q) + \chi_{[(n+1)N-j]}^{\rm NS}(q)]\end{aligned} \tag{9.14}$$

for even j. Using (9.11) and (9.12), this becomes

$$\begin{aligned}\chi^{\mathrm{c};\ell}_{[j]}(q)\,\chi^{(2)}_{[j]}(q) &= \chi^{\mathrm{NS}}_{[j]}(q)\sum_{n\in\mathbb{Z}} q^{nj+n^2N/2} \\ &= q^{-j^2/2N}\,\psi_{\ell,j}(q)\,\chi^{\mathrm{NS}}_{[j]}(q) = q^{-j^2/2N}\,\psi_{\ell,j}(q)\,\frac{\Theta_{N,j}(q)}{(\varphi(q))^N}\,.\end{aligned} \tag{9.15}$$

Analogously, with (7.9) we obtain the same result for odd j. By inserting the coset characters $\chi^{\mathrm{c};\ell}_{[j]}$ (7.2) we then get

$$\chi^{(2)}_{[j]}(q) = q^{-j^2/2N}\,\frac{\Theta_{N,j}(q)}{(\varphi(q))^{N-1}}\,. \tag{9.16}$$

For $j=\ell$ one has to read this result with (7.10), which means that our result only describes the sum of the irreducible characters $\chi^{(2)}_{\mathrm{s}}$ and $\chi^{(2)}_{\mathrm{c}}$. By comparison with (9.10), we may also rewrite the result in the form

$$\chi^{\mathrm{NS}}_{[nN+j]}(q) = \frac{q^{(nN+j)^2/2N}}{\varphi(q)}\,\chi^{(2)}_{[j]}(q)\,,\qquad \chi^{\mathrm{NS}}_{[(n+1)N-j]}(q) = \frac{q^{((n+1)N-j)^2/2N}}{\varphi(q)}\,\chi^{(2)}_{[j]}(q) \tag{9.17}$$

for $j=1,2,\dots,\ell$.

Comparing (6.9) again with (7.9), we also find

$$\begin{aligned}\chi^{\mathrm{c};\ell}_{\circ}(q)\,\chi^{(2)}_{\circ}(q)+\chi^{\mathrm{c};\ell}_{\mathrm{v}}(q)\,\chi^{(2)}_{\mathrm{v}}(q) &= \chi^{\mathrm{NS}}_{0}(q)+\sum_{n=1}^{\infty}\chi^{\mathrm{NS}}_{[nN]}(q) \\ &= \left[\chi^{\mathrm{NS}}_{0}(q)+\chi^{\mathrm{NS}}_{J}(q)\right]\left[\tfrac12+\tfrac12\,\psi_{\ell,0}(q)\right]-\chi^{\mathrm{NS}}_{J}(q) \\ &= \tfrac12\left[\chi^{\mathrm{NS}}_{0}(q)-\chi^{\mathrm{NS}}_{J}(q)\right]+\tfrac12\,\psi_{\ell,0}(q)\left[\chi^{\mathrm{NS}}_{0}(q)+\chi^{\mathrm{NS}}_{J}(q)\right] \\ &= \frac{(\varphi(q))^N}{2\,(\varphi(q^2))^N}+\psi_{\ell,0}(q)\,\frac{\Theta_{N,0}(q)}{2\,(\varphi(q))^N}\end{aligned} \tag{9.18}$$

and

$$\begin{aligned}\chi^{\mathrm{c};\ell}_{\circ}(q)\,\chi^{(2)}_{\mathrm{v}}(q)+\chi^{\mathrm{c};\ell}_{\mathrm{v}}(q)\,\chi^{(2)}_{\circ}(q) &= \chi^{\mathrm{NS}}_{J}(q)+\sum_{n=1}^{\infty}\chi^{\mathrm{NS}}_{[nN]}(q) \\ &= -\frac{(\varphi(q))^N}{2\,(\varphi(q^2))^N}+\psi_{\ell,0}(q)\,\frac{\Theta_{N,0}(q)}{2\,(\varphi(q))^N}\,.\end{aligned} \tag{9.19}$$

Subtraction of (9.19) from (9.18) yields

$$\left[\chi^{\mathrm{c};\ell}_{\circ}(q)-\chi^{\mathrm{c};\ell}_{\mathrm{v}}(q)\right]\cdot\left[\chi^{(2)}_{\circ}(q)-\chi^{(2)}_{\mathrm{v}}(q)\right] = \left[\tfrac{\varphi(q)}{\varphi(q^2)}\right]^N \equiv \chi^{\mathrm{NS}}_{0}(q)-\chi^{\mathrm{NS}}_{J}(q)\,, \tag{9.20}$$

so that by inserting (7.7) we obtain

$$\chi^{(2)}_{\circ}(q)-\chi^{(2)}_{\mathrm{v}}(q) = \left[\tfrac{\varphi(q)}{\varphi(q^2)}\right]^{N-1}. \tag{9.21}$$

Analogously, by adding (9.18) and (9.19) we get

$$\left[\chi^{\mathrm{c};\ell}_{\circ}(q)+\chi^{\mathrm{c};\ell}_{\mathrm{v}}(q)\right]\cdot\left[\chi^{(2)}_{\circ}(q)+\chi^{(2)}_{\mathrm{v}}(q)\right] = \psi_{\ell,0}(q)\,\frac{\Theta_{N,0}(q)}{(\varphi(q))^N}\,, \tag{9.22}$$

and hence inserting (7.8) we obtain

$$\chi_\circ^{(2)}(q) + \chi_{\rm v}^{(2)}(q) = \frac{\Theta_{N,0}(q)}{(\varphi(q))^{N-1}}\,. \tag{9.23}$$

In summary, we have derived that

$$\begin{aligned} \chi_\circ^{(2)}(q) &= \tfrac12\Big\{\tfrac{\Theta_{N,0}(q)}{(\varphi(q))^{N-1}} + \Big[\tfrac{\varphi(q)}{\varphi(q^2)}\Big]^{N-1}\Big\} \equiv \tfrac{1}{2\,(\varphi(q))^{N-1}}\,\big[\Theta_{N,0}(q) + (\psi_{1,0}(-q))^{N-1}\big]\,, \\ \chi_{\rm v}^{(2)}(q) &= \tfrac12\Big\{\tfrac{\Theta_{N,0}(q)}{(\varphi(q))^{N-1}} - \Big[\tfrac{\varphi(q)}{\varphi(q^2)}\Big]^{N-1}\Big\} \equiv \tfrac{1}{2\,(\varphi(q))^{N-1}}\,\big[\Theta_{N,0}(q) - (\psi_{1,0}(-q))^{N-1}\big]\,. \end{aligned} \tag{9.24}$$

Further, comparison with (9.7) and (9.9) yields

$$\chi_0^{\rm NS}(q) + \chi_J^{\rm NS}(q) = \frac{1}{\varphi(q)}\,[\chi_\circ^{(2)}(q) + \chi_{\rm v}^{(2)}(q)]\,, \tag{9.25}$$

while comparison with (9.10) and (9.11) shows that

$$\chi_{[nN]}^{\rm NS}(q) = \frac{q^{n^2N/2}}{\varphi(q)}\,[\chi_\circ^{(2)}(q) + \chi_{\rm v}^{(2)}(q)]\,. \tag{9.26}$$

9.3 $\widehat{\rm so}(N)_2$ characters for odd N

Now we consider the case $N = 2\ell + 1$. From (6.9) and (7.11) we find

$$\begin{aligned} \chi_{[2j]}^{{\rm c};2N}(q)\,\chi_{[j]}^{(2)}(q) &= \chi_{[j]}^{\rm NS}(q) + \chi_{[2N-j]}^{\rm NS}(q) + \chi_{[2N+j]}^{\rm NS}(q) + \chi_{[4N-j]}^{\rm NS}(q) + \chi_{[4N+j]}^{\rm NS}(q) + \dots \\ &\equiv \sum_{n=0}^{\infty}[\chi_{[2nN+j]}^{\rm NS}(q) + \chi_{[2(n+1)N-j]}^{\rm NS}(q)] \\ &= \chi_{[j]}^{\rm NS}(q)\sum_{n\in\mathbb{Z}} q^{2nj+2n^2N} = q^{-j^2/2N}\,\psi_{2N,2j}(q)\,\chi_{[j]}^{\rm NS}(q) \end{aligned} \tag{9.27}$$

for j even, and

$$\begin{aligned} \chi_{[2N-2j]}^{{\rm c};2N}(q)\,\chi_{[j]}^{(2)}(q) &= \sum_{n=0}^{\infty}[\chi_{[(2n+1)N+j]}^{\rm NS}(q) + \chi_{[(2n+1)N-j]}^{\rm NS}(q)] = \chi_{[j]}^{\rm NS}(q)\sum_{n\in\mathbb{Z}} q^{-(2n+1)j+(2n+1)^2N/2} \\ &= q^{-j+N/2}\chi_{[j]}^{\rm NS}(q)\sum_{n\in\mathbb{Z}} q^{2n(N-j)+2n^2N} = q^{-j^2/2N}\,\psi_{2N,2N-2j}(q)\,\chi_{[j]}^{\rm NS}(q) \end{aligned} \tag{9.28}$$

for j odd. By inserting the coset characters (7.2) we then arrive once again at the formulæ (9.16) and (9.17) for $j = 1, 2, \dots, \ell$.

In the same manner we find

$$\begin{aligned} \chi_\circ^{{\rm c};2N}(q)\,\chi_\circ^{(2)}(q) + \chi_{\rm v}^{{\rm c};2N}(q)\,\chi_{\rm v}^{(2)}(q) &= \chi_0^{\rm NS}(q) + \sum_{n=1}^{\infty}\chi_{[2nN]}^{\rm NS}(q) \\ &= [\chi_0^{\rm NS}(q) + \chi_J^{\rm NS}(q)]\,\Big[\tfrac12 + \tfrac12\sum_{n\in\mathbb{Z}} q^{2n^2N}\Big] - \chi_J^{\rm NS}(q) \\ &= \tfrac{(\varphi(q))^N}{2\,(\varphi(q^2))^N} + \psi_{2N,0}(q)\,\tfrac{\Theta_{N,0}(q)}{2\,(\varphi(q))^N} \end{aligned} \tag{9.29}$$

and

$$
\begin{aligned}
\chi_{\circ}^{\mathrm{c};2N}(q)\,\chi_{\mathrm{v}}^{(2)}(q)+\chi_{\mathrm{v}}^{\mathrm{c};2N}(q)\,\chi_{\circ}^{(2)}(q) &= \chi_J^{\mathrm{NS}}(q)+\sum_{n=1}^{\infty}\chi_{[2nN]}^{\mathrm{NS}}(q) \\
&= -\frac{(\varphi(q))^N}{2\,(\varphi(q^2))^N}+\psi_{2N,0}(q)\,\frac{\Theta_{N,0}(q)}{2\,(\varphi(q))^N}\,.
\end{aligned}
\tag{9.30}
$$

Thus we also obtain again the relations (9.21) and (9.23) for $\chi_{\circ}^{(2)}$ and $\chi_{\mathrm{v}}^{(2)}$, and hence also (9.24) and (9.26).

10 Summary and outlook

10.1 Decomposition of the tensor product

Let us now summarize some of our results on the tensor product decompositions. To this end we first note that $q^{\Delta}/\varphi(q)$ is precisely the character of the Verma module $M(c,\Delta)$ of the Virasoro algebra. For central charge $c=1$ the Verma module $M(c,\Delta)$ is irreducible as long as $4\Delta\neq m^2$ for $m\in\mathbb{Z}$; otherwise there exist null states. The characters of the irreducible modules $V(1,\Delta)$ of the $c=1$ Virasoro algebra are then given by

$$
\chi_{\Delta}^{\mathrm{Vir}}(q)=\begin{cases}(\varphi(q))^{-1}\,[q^{m^2/4}-q^{(m+2)^2/4}] & \text{if } \Delta=\frac{m^2}{4} \text{ with } m\in\mathbb{Z}\,,\\ (\varphi(q))^{-1}\,q^{\Delta} & \text{otherwise.}\end{cases}
\tag{10.1}
$$

Thus for $4\Delta=m^2$ with $m\in\mathbb{Z}$ the Verma module character can be decomposed as follows:

$$
\frac{q^{m^2/4}}{\varphi(q)}=\frac{1}{\varphi(q)}\sum_{k=0}^{\infty}\left[q^{(m+2k)^2/4}-q^{(m+2k+2)^2/4}\right]=\sum_{k=0}^{\infty}\chi_{(m+2k)^2/4}^{\mathrm{Vir}}(q)\,.
\tag{10.2}
$$

Correspondingly we write

$$
W(1,\Delta):=\begin{cases}\displaystyle\bigoplus_{k=0}^{\infty}V(1,\frac{(m+2k)^2}{4}) & \text{if } \Delta=\frac{m^2}{4} \text{ with } m\in\mathbb{Z}\,,\\ V(1,\Delta) & \text{otherwise.}\end{cases}
\tag{10.3}
$$

Using also the formulæ (8.29) and (8.30) for the coset conformal weights, we can summarize our results of section 9 by the following description of the big Fock space $\hat{\mathcal{H}}_{\mathrm{NS}}$. Recalling the decomposition

$$
\hat{\mathcal{H}}_{\mathrm{NS}}=\mathcal{H}_0\oplus\mathcal{H}_J\oplus\bigoplus_{m=1}^{\infty}(\mathcal{H}_{[m]}\otimes\mathbb{C}^2)
\tag{10.4}
$$

of $\hat{\mathcal{H}}_{\mathrm{NS}}$ into $\mathfrak{A}$-sectors, we can express the splitting of $\hat{\mathcal{H}}_{\mathrm{NS}}$ into tensor products of the Virasoro modules (10.3) and the irreducible highest weight modules of $\widehat{\mathrm{so}}(N)_2$ (that is, $\mathcal{H}_{\circ}^{(2)}$, $\mathcal{H}_{\mathrm{v}}^{(2)}$, $\mathcal{H}_{[j]}^{(2)}$, and also $\mathcal{H}_{\mathrm{s}}^{(2)}$ and $\mathcal{H}_{\mathrm{c}}^{(2)}$ when $N=2\ell$) as follows. Our results show that

$$
\mathcal{H}_{[nN]}=\left[\mathcal{H}_{\circ}^{(2)}\oplus\mathcal{H}_{\mathrm{v}}^{(2)}\right]\otimes\,W(1,\Delta_{n;\circ}^{\mathrm{c}})\,,
\tag{10.5}
$$

for $n = 1, 2, \dots$, as well as

$$\begin{aligned} \mathcal{H}_{[nN+j]} &= \mathcal{H}^{(2)}_{[j]} \otimes W(1, \Delta^{\rm c}_{n;j}) \,, \\ \mathcal{H}_{[(n+1)N-j]} &= \mathcal{H}^{(2)}_{[j]} \otimes W(1, \bar\Delta^{\rm c}_{n;j}) \end{aligned} \tag{10.6}$$

for $n = 0, 1, \dots$ and $j = 1, 2, \dots, \ell-1$. When $N = 2\ell+1$, (10.6) also holds for $j = \ell$, while for $j = \ell$ and $N = 2\ell$ we have

$$\mathcal{H}_{[nN+\ell]} = \left[\mathcal{H}^{(2)}_{\rm s} \oplus \mathcal{H}^{(2)}_{\rm c}\right] \otimes W(1, \Delta^{\rm c}_{n;{\rm s}}) \tag{10.7}$$

for $n = 0, 1, \dots$. Note that the modules $W(1, \Delta)$ appearing in the decompositions (10.5), (10.6) and (10.7) are all irreducible as long as $\sqrt{2N} \notin \mathbb{N}$. Otherwise we can write $N = 2K^2$ with $K \in \mathbb{N}$, and then the modules $W(1, \Delta^{\rm c}_{n;\circ})$ and $W(1, \Delta^{\rm c}_{n;j})$, $W(1, \bar\Delta^{\rm c}_{n;j})$ with $j = mK$, $m = 1, 2, \dots$ and $j \le \ell$, split up as in (10.3).

Besides the coset Virasoro generators, the chiral symmetry algebra of the orbifold coset theory contains further operators [17]. The observation above implies in particular that when acting on $\mathfrak{A}$-sectors other than $\mathcal{H}_0$ and $\mathcal{H}_J$, for $\sqrt{2N} \notin \mathbb{N}$ all these additional generators make transitions between the sectors of the gauge invariant fermion algebra $\mathfrak{A}$; for $N = 2K^2$ ($K \in \mathbb{N}$) the additional generators generically still make transitions, except that they can map sectors with $j = mK$ to themselves. It follows in particular that we can distinguish between elements of the coset Virasoro algebra and elements of the full coset chiral algebra which are not contained in the coset Virasoro algebra by acting with them on suitable $\mathfrak{A}$-sectors.

10.2 The sectors $\mathcal{H}_0$ and $\mathcal{H}_J$

It still remains to analyze the decomposition of the $\mathfrak{A}$-sectors $\mathcal{H}_0$ and $\mathcal{H}_J$ explicitly. From (9.25) we conclude that

$$\mathcal{H}_0 \oplus \mathcal{H}_J = \left[\mathcal{H}^{(2)}_\circ \oplus \mathcal{H}^{(2)}_{\rm v}\right] \otimes W(1, 0) \,. \tag{10.8}$$

Now $W(1, 0)$ is always reducible, independent of the particular value of the integer N. We claim that

$$\begin{aligned} \mathcal{H}_0 &= \mathcal{H}^{(2)}_\circ \otimes \bigoplus_{k=0}^\infty V(1, (2k)^2) \;\oplus\; \mathcal{H}^{(2)}_{\rm v} \otimes \bigoplus_{k=0}^\infty V(1, (2k+1)^2) \,, \\ \mathcal{H}_J &= \mathcal{H}^{(2)}_\circ \otimes \bigoplus_{k=0}^\infty V(1, (2k+1)^2) \;\oplus\; \mathcal{H}^{(2)}_{\rm v} \otimes \bigoplus_{k=0}^\infty V(1, (2k)^2) \,. \end{aligned} \tag{10.9}$$

This can be seen by decomposing the characters $\chi_0^{\rm NS}$ and $\chi_J^{\rm NS}$ as follows:

$$
\begin{aligned}
\chi_0^{\rm NS}(q) &= \frac{\Theta_{N,0}(q)}{2(\varphi(q))^N} + \frac{(\varphi(q))^{N-2}}{2(\varphi(q^2))^{N-1}} \sum_{k\in\mathbb{Z}} (-1)^k q^{k^2} \\
&\equiv \frac{\Theta_{N,0}(q)}{2(\varphi(q))^N} + \frac{(\varphi(q))^{N-2}}{2(\varphi(q^2))^{N-1}} \sum_{k=0}^{\infty} \left[q^{(2k)^2} - 2q^{(2k+1)^2} + q^{(2k+2)^2}\right] \\
&= \chi_\circ^{(2)}(q) \frac{1}{\varphi(q)} \sum_{k=0}^{\infty} \left[q^{(2k)^2} - q^{(2k+1)^2}\right] + \chi_{\rm v}^{(2)}(q) \frac{1}{\varphi(q)} \sum_{k=0}^{\infty} \left[q^{(2k+1)^2} - q^{(2k+2)^2}\right] \\
&\equiv \chi_\circ^{(2)}(q) \cdot \sum_{k=0}^{\infty} \chi^{\rm Vir}_{(2k)^2}(q) + \chi_{\rm v}^{(2)}(q) \cdot \sum_{k=0}^{\infty} \chi^{\rm Vir}_{(2k+1)^2}(q)\,.
\end{aligned}
\tag{10.10}
$$

(In the first line we used (7.6).) Similarly,

$$
\chi_J^{\rm NS} = \chi_\circ^{(2)} \cdot \sum_{k=0}^{\infty} \chi^{\rm Vir}_{(2k+1)^2} + \chi_{\rm v}^{(2)} \cdot \sum_{k=0}^{\infty} \chi^{\rm Vir}_{(2k)^2}\,. \tag{10.11}
$$

It follows that besides $|\Omega_\circ^{0,0}\rangle \equiv |\Omega\rangle$ and $|\Omega_{\rm v}^{J,0}\rangle \equiv |\Omega_{\rm v}\rangle$, there must exist further simultaneous highest weight vectors of $\widehat{\rm so}(N)_2$ and the coset Virasoro algebra, namely, for $k = 0, 1, 2, \ldots$, highest weight vectors $|\Omega_\circ^{0,2k+2}\rangle$, $|\Omega_{\rm v}^{0,2k+1}\rangle \in \mathcal{H}_0$ and $|\Omega_\circ^{J,2k+1}\rangle$, $|\Omega_{\rm v}^{J,2k+2}\rangle \in \mathcal{H}_J$, with $\widehat{\rm so}(N)_2$-weights $\Lambda_\circ$, $\Lambda_{\rm v}$, $\Lambda_\circ$, $\Lambda_{\rm v}$, respectively, and with coset conformal weights $(2k+2)^2$, $(2k+1)^2$, $(2k+1)^2$, $(2k+2)^2$, respectively. Those vectors with unit coset conformal weight have a relatively simple form; we find

$$
|\Omega_\circ^{J,1}\rangle = \begin{cases} \displaystyle\sum_{k=1}^{\ell} \left(\bar x^{k,+}_{-1/2} x^{k,-}_{-1/2} - \bar x^{k,-}_{-1/2} x^{k,+}_{-1/2}\right) |\Omega\rangle & \text{for } N = 2\ell\,, \\ \displaystyle\Big\{ \sum_{k=1}^{\ell} \left(\bar x^{k,+}_{-1/2} x^{k,-}_{-1/2} - \bar x^{k,-}_{-1/2} x^{k,+}_{-1/2}\right) + \bar x^{\ell+1,+}_{-1/2}\, \bar x^{\ell+1,-}_{-1/2} \Big\} |\Omega\rangle & \text{for } N = 2\ell+1\,, \end{cases} \tag{10.12}
$$

as well as

$$
|\Omega_{\rm v}^{0,1}\rangle = x^{1,+}_{-1/2} x^{1,-}_{-1/2}\, |\Omega_\circ^{J,1}\rangle + (x^{1,+}_{-3/2} x^{1,-}_{-1/2} + x^{1,-}_{-3/2} x^{1,+}_{-1/2})\, |\Omega\rangle\,. \tag{10.13}
$$

In contrast, the highest weight vectors with larger coset conformal weight are more difficult to identify.

10.3 $\widehat{\rm so}(N)_2$ characters

Our idea to employ the representation theory of the gauge group O(2) allowed us to deduce simple formulæ for the characters of the (Neveu–Schwarz sector) irreducible highest weight modules of $\widehat{\rm so}(N)$ at level two. They are given by the expressions (9.16) for $\chi^{(2)}_{[j]}$ and (9.24) for $\chi_\circ^{(2)}$ and $\chi_{\rm v}^{(2)}$. Note that, not surprisingly, these results have a simple functional dependence on the integer N, even though the details of their derivation (involving e.g. the relation with the orbifold coset theory) depend quite non-trivially on whether N is even or odd.

Our results for these characters are not new. In [12], the conformal embedding of $\widehat{\text{so}}(N)_2$ into $\widehat{\text{u}}(N)$ at level one was employed to identify (sums of) $\widehat{\text{so}}(N)_2$ characters with characters of $\widehat{\text{su}}(N)_1$. Indeed, the restricted summation over the lattice vector $\mathbf{m} \in \mathbb{Z}^N$ in the formula (9.8) for $\Theta_{N,m}(q)$ precisely corresponds to the summation over the appropriately shifted root lattice of su(N).

With the help of the conformal embedding only the linear combination $\chi^{(2)}_\circ + \chi^{(2)}_{\rm v}$ of the irreducible characters $\chi^{(2)}_\circ$ and $\chi^{(2)}_{\rm v}$ is obtained, which is just the level-one vacuum character of $\widehat{\text{su}}(N)$. However, the orthogonal linear combination $\chi^{(2)}_\circ - \chi^{(2)}_{\rm v}$ is known as well; it has been obtained in [11, p. 233] by making use of the theory of modular forms.

10.4 A homomorphism of fusion rings

In section 8 we were able to identify the $\widehat{\text{so}}(N)_2$ highest weight modules within the sectors of the intermediate algebra $\mathfrak{A}$ which are governed by the gauge group O(2). Our results amount to the following assignment ρ of the O(2)-representations Φ to the WZW sectors ϕ:

$$\begin{aligned}
&\rho(\Phi_0) = \phi_\circ\,, \qquad \rho(\Phi_J) = \phi_{\rm v}\,,\\
&\rho(\Phi_{[(n+1)N]}) = \phi_\circ + \phi_{\rm v}\,,\\
&\rho(\Phi_{[nN+j]}) = \rho(\Phi_{[(n+1)N-j]}) = \phi_{[j]} \qquad \text{for } j=1,2,\ldots,\ell-1\,,\\
&\rho(\Phi_{[nN+\ell]}) = \rho(\Phi_{[(n+1)N-\ell]}) = \begin{cases} \phi_{\rm s}+\phi_{\rm c} & \text{for } N=2\ell\,,\\ \phi_{[\ell]} & \text{for } N=2\ell+1\,, \end{cases}
\end{aligned} \tag{10.14}$$

for $n = 0, 1, 2, \ldots$ (Note that in the case of Φ_0 and Φ_J, the action of ρ does not directly correspond to the decomposition of the $\mathfrak{A}$-sectors into $\widehat{\text{so}}(N)_2$ sectors.)

The multiplication rules of the representation ring $\mathcal{R}_{\rm O(2)}$ of O(2) are given by the relations (4.8). The level-two WZW sectors generate a fusion ring, too, which we denote by $\mathcal{R}^{(2)}_{\rm WZW}$. The ring $\mathcal{R}^{(2)}_{\rm WZW}$ has a fusion subring $\mathcal{R}^{(2)}_{\rm NS}$ which is generated by those primary fields which appear in the Neveu–Schwarz sector $\hat{\mathcal{H}}_{\rm NS}$. The fusion rules, i.e. the structure constants of $\mathcal{R}^{(2)}_{\rm WZW}$, can be computed with the help of the Kac–Walton and Verlinde formulæ (see e.g. [18]). For the subring $\mathcal{R}^{(2)}_{\rm NS}$ one finds the tensor product decompositions listed in appendix A.3.

Inspection shows that $\mathcal{R}^{(2)}_{\rm NS}$ is in fact isomorphic to the representation ring of the dihedral group $\mathcal{D}_N$. Now for any N the group $\mathcal{D}_N$ is a finite subgroup of O(2). As a consequence, the mapping ρ actually constitutes a fusion ring *homomorphism* from the representation ring $\mathcal{R}_{\rm O(2)}$ of O(2) to the fusion subring $\mathcal{R}^{(2)}_{\rm NS}$ of $\mathcal{R}^{(2)}_{\rm WZW}$. (It is also easily checked that for odd N the homomorphism ρ is surjective, while for even N the image does not contain the linear combination $\phi_{\rm s} - \phi_{\rm c}$.) This observation explains to a certain extent why, in spite of the fact that the WZW observable algebra $\mathfrak{A}_{\rm WZW}$ is much smaller than the O(2)-invariant algebra $\mathfrak{A}$, the group O(2) nevertheless provides a substitute for the gauge group in the DHR sense. But even in view of this relationship it is still surprising how closely the WZW superselection structure follows the representation theory of O(2).

One may speculate that the presence of the homomorphism ρ indicates that the gauge group O(2) is in fact part of the full (as yet unknown) quantum symmetry of the WZW theory that fully takes over the rôle of the DHR gauge group. This is possible because all sectors in the Neveu–Schwarz part of the WZW theory have integral quantum dimen-

sion. Now in rational conformal field theory sectors with integral quantum dimension are actually extremely rare. It will be interesting to study the relationship between the representation ring of $\mathrm{O}(k^\vee)$ or $\mathrm{U}(k^\vee)$ and the WZW fusion ring in more general cases where (most of) the WZW sectors possess non-integral quantum dimension.

A Appendix

A.1 Commutators of the fermion modes $x_r^{k,\pm}$ with the currents

By direct calculation, we obtain

$$[\mathcal{H}^j, x_r^{k,\pm}] = \delta_{j,k}\, x_r^{k,\pm}\,, \qquad [\mathcal{H}^j, \bar{x}_r^{k,\pm}] = -\delta_{j,k}\, \bar{x}_r^{k,\pm}\,, \tag{A.1}$$

for all $j, k = 1, 2, \ldots, \ell$, and similarly, for $N = 2\ell + 1$,

$$[\mathcal{H}^j, \bar{x}_r^{\ell+1,\pm}] = 0 \tag{A.2}$$

for all $j = 1, 2, \ldots, \ell$.

To find also the commutators of the fermion modes with the raising operators $\mathcal{E}_+^j$, we first compute

$$[J_m(t_{\varepsilon,\eta}^{i,j}), c_r^{k,\pm}] = \frac{1}{2}\,\varepsilon\,(\eta \mp 1)\,\delta_{j,k}\, c_{m+r}^{i,\varepsilon} - \frac{1}{2}\,\eta\,(\varepsilon \mp 1)\,\delta_{i,k}\, c_{m+r}^{j,\eta}\,. \tag{A.3}$$

Analogous relations hold for $[J_m(t_{\varepsilon,\eta}^{i,j}), \bar{c}_r^{k,\pm}]$. When $N = 2\ell + 1$ we have in addition the relation $[J_m(t_{\varepsilon,\eta}^{i,j}), b_r^{2\ell+1;q}] = 0$ and

$$\begin{aligned} &[J_m(t_+^j), c_r^{k,\pm}] = \mp\,\delta_{j,k}\, b_{m+r}^{2\ell+1;1}\,, \qquad [J_m(t_-^j), c_r^{k,\pm}] = 0\,, \\ &[J_m(t_\pm^j), b_r^{2\ell+1;1}] = -\, c_{m+r}^{j,\pm}\,, \end{aligned} \tag{A.4}$$

and similar relations for $[J_m(t_+^j), \bar{c}_r^{k,\pm}]$, $[J_m(t_-^j), \bar{c}_r^{k,\pm}]$ and $[J_m(t_\pm^j), b_r^{2\ell+1;2}]$. From these results we learn that

$$\begin{aligned} &[\mathcal{E}_+^0, x_r^{k,\pm}] = \delta_{k,2}\,\bar{x}_{r+1}^{1,\pm} - \delta_{k,1}\,\bar{x}_{r+1}^{2,\pm}\,, \qquad [\mathcal{E}_+^0, \bar{x}_r^{k,\pm}] = 0\,, \\ &[\mathcal{E}_+^j, x_r^{k,\pm}] = -\delta_{k,j+1}\, x_r^{j,\pm}\,, \qquad [\mathcal{E}_+^j, \bar{x}_r^{k,\pm}] = \delta_{k,j}\,\bar{x}_r^{j+1,\pm} \quad \text{for } j = 1, 2, \ldots, \ell - 1\,, \\ &[\mathcal{E}_+^\ell, x_r^{k,\pm}] = 0\,, \qquad [\mathcal{E}_+^\ell, \bar{x}_r^{k,\pm}] = \begin{cases} \delta_{k,\ell}\, x_r^{\ell-1,\pm} - \delta_{k,\ell-1}\, x_r^{\ell,\pm} & \text{for } N = 2\ell\,, \\ \delta_{k,\ell}\,\bar{x}_r^{\ell+1,\pm} - \delta_{k,\ell+1}\, x_r^{\ell,\pm} & \text{for } N = 2\ell + 1\,. \end{cases} \end{aligned} \tag{A.5}$$

Taking into account that $(x_r^{j,\pm})^2 = (\bar{x}_r^{j,\pm})^2 = 0$ and $(x_r^{j,\pm})^* = \bar{x}_{-r}^{j,\mp}$, these relations imply that

$$\begin{aligned} &[\mathcal{E}_+^j, X_r^{k,\pm}] = 0 \qquad \text{for } j = 1, 2, \ldots, \ell\,, \\ &[\mathcal{E}_+^j, \bar{X}_r^{k,\pm}] = 0 \qquad \text{for } j = 1, 2, \ldots, \ell - 1\,. \end{aligned} \tag{A.6}$$

For $j = \ell$ we have instead

$$\begin{aligned} &[\mathcal{E}_+^\ell, \bar{X}_r^{k,\pm}] \cdot X_r^{\ell,\pm} = 0 \qquad \text{for } N = 2\ell\,, \\ &[\mathcal{E}_+^\ell, \bar{X}_r^{k,\pm}] \cdot \bar{x}_r^{\ell+1,\pm} = 0 \quad \text{and} \quad [\mathcal{E}_+^\ell, \bar{x}_r^{\ell+1,\pm}] \cdot X_r^{\ell,\pm} = 0 \quad \text{for } N = 2\ell + 1\,. \end{aligned} \tag{A.7}$$

Finally, for $j = 0$ we find

$$[\mathcal{E}_+^0, \bar{X}_r^{k,\pm}] = 0\,, \qquad [\mathcal{E}_+^0, X_r^{k,\pm}] \cdot \bar{X}_{r+1}^{0,\pm} = 0\,. \tag{A.8}$$

A.2 The action of the gauge group O(2)

For the Fourier modes $c_r^{j,\pm}$ and $\bar{c}_r^{j,\pm}$ the actions (4.2) of γ_t, $t\in\mathbb{R}$, and (4.3) of η read

$$\begin{aligned} &\gamma_t(c_r^{j,\pm}) = \cos(t)\, c_r^{j,\pm} - \sin(t)\, \bar{c}_r^{j,\pm}\,, \quad &&\gamma_t(\bar{c}_r^{j,\pm}) = \sin(t)\, c_r^{j,\pm} + \cos(t)\, \bar{c}_r^{j,\pm}\,, \\ &\eta(c_r^{j,\pm}) = c_r^{j,\pm}\,, &&\eta(\bar{c}_r^{j,\pm}) = -\bar{c}_r^{j,\pm}\,, \end{aligned} \tag{A.9}$$

so that the combinations $x_r^{j,\pm}$ transform as

$$\gamma_t(x_r^{j,\pm}) = \mathrm{e}^{\pm\mathrm{i}t}\, x_r^{j,\pm}\,, \quad \eta(x_r^{j,\pm}) = x_r^{j,\mp}\,. \tag{A.10}$$

Analogously,

$$\gamma_t(\bar{x}_r^{j,\pm}) = \mathrm{e}^{\pm\mathrm{i}t}\, \bar{x}_r^{j,\pm}\,, \quad \eta(\bar{x}_r^{j,\pm}) = \bar{x}_r^{j,\mp}\,. \tag{A.11}$$

Hence the combinations $X_r^{j,\pm}$ transform as

$$\gamma_t(X_r^{j,\pm}) = \mathrm{e}^{\pm\mathrm{i}jt}\, X_r^{j,\pm}\,, \quad \eta(X_r^{j,\pm}) = X_r^{j,\mp}\,, \tag{A.12}$$

and analogously,

$$\gamma_t(\bar{X}_r^{j,\pm}) = \mathrm{e}^{\pm\mathrm{i}(\ell-j)t}\, \bar{X}_r^{j,\pm}\,, \quad \eta(\bar{X}_r^{j,\pm}) = \bar{X}_r^{j,\mp}\,. \tag{A.13}$$

A.3 The fusion rules of $\widehat{\mathrm{so}}(N)_2$

In this appendix we present the relations of the fusion ring $\mathcal{R}_{\mathrm{NS}}^{(2)} \subset \mathcal{R}_{\mathrm{WZW}}^{(2)}$, i.e. the fusion rules for those primary fields of the WZW theory based on $\widehat{\mathrm{so}}(N)_2$ which correspond to the $\widehat{\mathrm{so}}(N)_2$ highest weight modules that appear in the Neveu–Schwarz sector. For $N = 2\ell$ we have

$$\begin{aligned} &\phi_\mathrm{v} \star \phi_\mathrm{v} = \phi_\circ\,, \qquad \phi_\mathrm{v} \star \phi_\mathrm{s} = \phi_\mathrm{c}\,, \\ &\phi_\mathrm{s} \star \phi_\mathrm{s} = \phi_\mathrm{c} \star \phi_\mathrm{c} = \begin{cases} \phi_\circ & \text{for } \ell\in 2\mathbb{Z}\,, \\ \phi_\mathrm{v} & \text{for } \ell\in 2\mathbb{Z}+1\,, \end{cases} \\ &\phi_\mathrm{s} \star \phi_\mathrm{c} = \begin{cases} \phi_\mathrm{v} & \text{for } \ell\in 2\mathbb{Z}\,, \\ \phi_\circ & \text{for } \ell\in 2\mathbb{Z}+1\,, \end{cases} \\ &\phi_\mathrm{v} \star \phi_{[j]} = \phi_{[j]}\,, \qquad \phi_\mathrm{s} \star \phi_{[j]} = \phi_\mathrm{c} \star \phi_{[j]} = \phi_{[\ell-j]}\,, \\ &\phi_{[i]} \star \phi_{[j]} = \phi_{[\,|i-j|\,]} + \phi_{[i+j]}\,. \end{aligned} \tag{A.14}$$

Here it is to be understood that whenever on the right hand side a label j appears which is larger than ℓ, it must be interpreted as the number

$$j' := N - j\,, \tag{A.15}$$

and when the label equals zero or ℓ, one has to identify $\phi_{[j]}$ as the sum

$$\phi_{[0]} \equiv \phi_\circ + \phi_\mathrm{v}\,, \qquad \phi_{[\ell]} \equiv \phi_\mathrm{s} + \phi_\mathrm{c}\,. \tag{A.16}$$

For $N = 2\ell+1$ the fusion rules read

$$\begin{aligned} &\phi_\mathrm{v} \star \phi_\mathrm{v} = \phi_\circ\,, \qquad \phi_\mathrm{v} \star \phi_{[j]} = \phi_{[j]}\,, \\ &\phi_{[i]} \star \phi_{[j]} = \phi_{[\,|i-j|\,]} + \phi_{[i+j]}\,. \end{aligned} \tag{A.17}$$

This time it is understood that when j is larger than ℓ, it stands for the number $j' := N-j$, and again that $\phi_{[0]} \equiv \phi_\circ + \phi_\mathrm{v}$.

The Neveu–Schwarz sector fusion rules which are not listed explicitly all follow from the commutativity and the associativity of the fusion product and from the fact that $\phi_\circ$ is the unit of the fusion ring.

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